MATHEMATICAL PROBLEMS FOR THE GIFTED: THE STRUCTURE OF PROBLEM SETS

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This paper discusses the structures of problem sets from textbooks for the mathematically gifted and suggests ways to analyze these structures. It is argued that seemingly purely technical details, such as the presence and quantity of simple problems or the grouping of problems, reflect important aspects in the understanding of gifted education, and more broadly, of the education of all students. An analysis of existing textbooks along the lines proposed here, it is argued, is useful for working teachers and textbook authors. But it may also interest a wider circle of mathematics educators, making it possible to pose a series of questions that may lead to new research.

Keywords: textbooks, gifted students, problems, problem sets, schools with an advanced course in mathematics.

INTRODUCTION

This article is part of a large study of Russian mathematics textbooks in which I am currently engaged. The analysis of textbooks and their history constitutes a complex methodological problem (Schubring, 1987). It is notable, however, that researchers usually focus on the theoretical material presented in a textbook, as it were, while devoting less attention to problems. In part, this is explained by the fact that problems were quite slow to appear in textbooks in general. On the other hand, even today, despite the fact that the importance of problem solving has become universally recognized, problems in textbooks are often seen as mere exercises, whose aim is to develop various skills, and which are consequently more or less typical and traditional, and therefore of little interest for discussion.

Consequently, the principles (methodology) for analyzing problem sets are also insufficiently developed. The analysis of problem sets is sometimes confined to simplistic quantitative parameters (how many problems are given per topic) and in the best cases to indications of different types of assignments -- multiple choice, short answer, or essay question. Although no theoretical generalization of methodological approaches is offered in the paper, examples of deeper analysis will be demonstrated.

Here, we will briefly discuss the distinctive features of the structure of problem sets from textbooks for the mathematically gifted (or from special problem books for these students). For this category of students, problems are especially important (although they are, of course, important for all other students as well). The discussion will focus on details that may be regarded as technical, but which are founded, as we will argue, on important considerations about the philosophy and organization of education. Note that the present study is not yet completed and represents only a relatively small part of the research being conducted, or more precisely, offers only certain examples of this research.

A BIT OF HISTORY

Let us state in somewhat greater detail which books will be analyzed and under what conditions these books are used. At the end of the 1950s, so called specialized mathematics schools appeared in the USSR (Karp, 2011; Vogeli, 1968; 1997). Sometimes these schools are called schools for the mathematically gifted, and this is justifiable, since many of their students have belonged to this category; however, it seems more correct to refer to them more neutrally as schools with an advanced course in mathematics.

These schools built their own curricula and traditions of teachings with the participation of leading Russian mathematicians, including Kolmogorov, Gelfand, Smirnov, Lavrentyev, and many others. Students were admitted to these schools on a competitive basis, often being selected from a whole large city or region. From the 1960s on, textbooks for teaching in these schools began to appear in one or another format.

Later, the number of such schools grew substantially. Special textbooks and problem book for such schools began to be published, sometimes in quite large numbers (for example, Alexandrov, Werner, Ryzhik, 2006a, b; Vilenkin, Ivashev-Musatov, Shvartsburd, 1995a, b; Galitsky, Goldman, Zvavitch, 1997; Karp, 2006). While at the beginning of their existence schools with an advanced course in mathematics included only the two upper grades, later a four-year course of advanced study, encompassing grades 8 to 11, became widespread.

It must be said that the curricula of such schools, whether provided by the Ministry of Education or formulated in the schools themselves, usually include many advanced sections that are either not studied in ordinary schools at all or studied there in incomparably less depth and detail. Below, we will subject those sections to analysis which at least in name correspond to those studied in ordinary schools. This will make it easier for us to draw comparisons and contrasts. Note that aside from textbooks and standard problem books, there also exists a vast supplementary literature for mathematics schools, which includes, for example, Olympiad problem books. These will not be analyzed below.

PROBLEM BOOKS FOR GRADES 8-9

Let us look in more detail at one of most widely used problem books in schools with an advanced course in mathematics: the problem book for grades 8-9 by Galitsky et al. (1997). This text began to be published in the mid 1990s and has gone through many editions since. Reflecting the curriculum that existed at the time of its publication, the problem book contains the following sections:

> Review; Divisibility of integers; Square roots; Quadratic equations; Inequalities; Powers with integer exponents; Functions; Equations and systems of equations; Word problems; Powers with rational exponents; Sequences and progressions; Trigonometric expressions and their transformations.

Practically all of these topics (except perhaps the divisibility of integers) were covered in one way or another at certain points in ordinary schools as well. We will see many more distinctive features, however, if we compare separate groups of problems. Consider, for example, the small and auxiliary section "Incomplete quadratic equations."

This section contains 16 assignments, each of which except the last four includes several—from three to six—problems. The assignments united under one number are not necessarily identical. For example, the first of them (5.1.) contains incomplete quadratic equations of the form $ax^2 = b$ with coefficients of different types (fractional, negative, etc.) as well as equations that are reducible to them, including equations that may be reduced to incomplete quadratic equations by introducing a new variable (for example, $4 - 9(2 - 5x)^2 = 0$). The remaining assignments are devoted to the following questions:

> Solving incomplete equations of the form $ax^2 + bx = 0$ and equations reducible to such equations; Incomplete equations with letter coefficients; Rational equations reducible to incomplete quadratic equations; Incomplete equations with absolute value notation; Problems on investigating equations with parameters; Word problems.

For comparison, consider an analogous section from the textbook by Dorofeev et al. (1999), written somewhat later than the problem book just cited and for ordinary schools. The problems in this textbook are divided into two groups, A and B, the second of which is addressed to stronger students. Section A contains 14 assignments, half of which contain six problems each (this time, the problems are, if not completely analogous to one another, then very similar). These problems are devoted to solving various types of incomplete equations and equations that are reducible to them. We should note the inclusion of problems—nonstandard for such a section—in which students are asked to solve incomplete cubic equations on the model of incomplete quadratic equations (for example, $x^3 - x = 0$). The remaining problems in this section are word problems.

Section B contains 12 assignments, which include equations that are reducible to incomplete quadratic equations, including those that may be reduced to incomplete quadratic equations by introducing a new variable, solving equations with letter coefficients, investigating equations with letter coefficients, and once again, word problems.

It may be noted at once that even if the topics in the two problem book are the same, a problem from Galitsky et al. (1997) is usually more difficult, if only in the sense that its solution requires more steps. Compare, for example, problem 490 (b) from Dorofeev et al. (1999) and 5.3. (d) from Galitsky et al. (1997). In the former, students are asked to solve the equation $ax^2 - x = 0$; in the latter, the equation $a(x^2 - 6x + 9) + 4 = 0$. Another factor, however, is probably even more important.

Even in comparing the problem book by Galitsky et al. (1997) with the text by Dorofeev et al. (1999), which is also written in such a way as to allow students who are gifted and interested in mathematics to use it, we can see that the differences between them consist not only in the technical difficulty of the problems, but in the different principles underlying their organization. The following three distinctive characteristics may be identified in books for mathematically advanced students:

- Fewer problems in which students are asked to directly apply and practice employing an algorithm that they have studied;
- Greater attention to developing some idea or transferring it onto other objects;
- More connections with other, previously studied sections.

BOOKS FOR GRADES 10-11

Textbooks for grades 10-11 with an advanced course in mathematics, as has already been said, contain many extra sections by comparison with ordinary textbooks. (For example, Vilenkin et al., 1995a, 1995b contains sections on limits and continuity, complex numbers and operations with complex numbers, elementary combinatorics and elementary theory of probability, while the textbook by Alimov et al., 1996 does not cover any of these topics.) Let us again compare sections that are present in both types of textbooks. For example, solving exponential equations and inequalities.

This section in the textbook by Vilenkin et al., 1995b is very short. Students are given only 14 equations and inequalities (arranged in two groups). The textbook by Alimov et al., 1996 offers 107 problems arranged in 26 groups. A comparison between these numbers, of course, tells us little if we do not also take into account the number of problems in each textbook overall (the textbook for ordinary schools also has more problems in general).

It is noteworthy, however, that in Alimov et al., 1996 the problems are arranged in such a way as to be next to problems that are very similar to them conceptually (within the same group). Sometimes, there is a certain development within a group. For example, group № 15 begins with the equation $3 \cdot 9^x = 81$ and ends with the equation $6^{3x-1} = 6^{1-2x}$. In both cases, naturally, the solution is based on the use of the fact that if $a^b = a^c$ (where a is a positive number not equal to one), then $b = c$. Nonetheless, the assignments are not identical.

They differ, however, far less than the assignments in the corresponding group in Vilenkin et al., 1995b. Here, the first equation is $4^{x-1}+4^{x}=320$, but already the third equation is $2\cdot 3^{x+1}-5\cdot 9^{x-2}=81$, in other words, an equation of a completely different type. Note also that Vilenkin et al., 1995b contains no completely elementary equations or inequalities such as $a^{x}=a^{b}$ or $a^{x}>a^{b}$ at all, while Alimov et al., 1996 contains quite a number of them.

On the other hand, Vilenkin et al., 1995b does contain assignments that connect the topic being studied with topics studied earlier. These include, for example, the following, in our view rather artificial, equation and inequality: $5^{2+4+...+2x}=0.04^{-45}$ and $\left|2^{x}-2\right|-\left|2^{x}-1\right|\geq\left|2^{x}+1\right|-5$, in order to solve which students must know how to find the sum of a geometric progression and to solve modulus inequalities. In other words, the distinctive characteristics that we noted earlier are present here as well.

DISCUSSION

Above, we deliberately discussed two typical algorithmic (technical, computational) sections. Neither solving quadratic equations nor solving exponential equations and inequalities can be considered topics that throw into some special relief the kind of conceptual understanding and depth that we would like to regard as typical manifestations of mathematical giftedness.

Of course, it is not difficult to cite examples from other topics, or problems characteristic of another way of organizing instruction. Probably the most vivid example of such a special way of organizing instruction is the system of instruction used in several Moscow schools that makes use of "leaflets" or "sheets" (Davidovich, Pushkar’, Chekanov, 2008). The problems used in this system of instruction are selected in such a way as to allow students themselves to arrive at the recognition of the fundamental theoretical principles of the course being studied, deriving and proving the corresponding propositions on their own.

In the West, a similar system has been called the "Texas method" or the "Moore method," since it is often associated with R.L.Moore, who made use of it (Parker, 2005). The fact that this system may be used only with highly gifted students, and even then only under special conditions, is obvious (for example, in Moscow's school № 57, at a class being conducted according to the system just described, in addition to the teacher-supervisor, there are four or five teaching assistants to whom students present their solutions, who clarify certain details, etc.). The discussion above, however, concerned no special sections and no special conditions.

The three distinctive characteristics of the structuring of problems in the educational literature for the gifted that we have identified need to be theoretically interpreted and viewed in the light of other theoretical insights into the gifted. The distinctive characteristics of the thinking of the mathematically gifted were already identified by

Krutetskii (1976), who took into account the opinions of practicing teachers. It may be said that practicing authors of textbooks rely on the following notions about the thinking of the gifted (which are reflected in the distinctive characteristics of the structures of problem sets discussed above):

- Gifted students grasp rules and algorithms more quickly and retain them better;
- Gifted students more easily make connections with other sections of mathematics that they have studied and are studying, more freely transferring what they have learned to other fields.

Here, we might ask to what degree these considerations (which largely parallel those formulated by Krutetskii) are borne out not only by practical experience, but also by organized studies. Let us note that the practical question concerning the number of assignments-exercises that students should be given for the purpose of practicing the application of some rule deserves serious investigation in general. It is not difficult to point to situations in which students are given literally hundreds of identical assignments so that they might memorize rules sufficiently thoroughly. On the other hand, one may also cite textbooks that offer only one-two identical problems. Of course, sometimes this is done because the authors in principle do not approve of memorization and consequently do not set themselves the task of facilitating it. But sometimes this is not the case—the authors think that certain things do have to be memorized thoroughly, but believe that even two same-type, one-step problems will suffice, while the rest will be learned in the course of working on more difficult assignments. Whether this is the case, or whether it pertains only to mathematically gifted students, or even whether this is not always true even for these students, or whether the outcome depends not on giftedness but on some other facts, are questions that, in our opinion, have not been sufficiently investigated.

In another paper (Karp, 2007), we discussed the notion that possession of great knowledge is evidence of talent. As can be seen, the authors of textbooks assume that gifted schoolchildren remember more than ordinary ones, and boldly weave into problems materials covered earlier. On the other hand, one could say that authors do this because they make it their goal to get schoolchildren to remember more than is demanded of them under ordinary conditions (and as experience shows, they achieve this goal). It would be interesting to collect data (experimental or drawn from teaching experience) about the success—or conversely, failure—of such an approach with "ordinary" students. What happens if they are treated in the same way as those who are recognized as gifted?

In the assignments analyzed above, individual ideas are developed only to a modest extent (which cannot be compared, for example, with the way in which ideas are developed in the aforementioned "sheets"), although this still goes beyond what is found in "ordinary" textbooks. More generally, shifts from one group of assignments to another are usually based on rather simple considerations—usually, they are due simply to the fact that another topic is being covered, that is, another type of assignment and another algorithm for solving such assignments. In another paper (Karp, 2002), we pointed out that the structure of problem material may be quite complex and discussed the importance of studying the "morphology of the problem set." It is noteworthy that, despite the virtually universal recognition accorded to the importance of creating opportunities for independent discovery by the students, in practice such opportunities, which are opened up by the nonstandard structuring of problem sets, are not very numerous in textbooks. This points to interesting directions for improvements in working both with gifted and with interested students, and, indeed, with all other students as well.

CONCLUSION

The practice and theory of working with mathematically gifted students are, we would argue, not infrequently divorced from one another (which happens often in mathematics education in general). Working teachers often do not know about the work of psychologists, although it would be useful for them, while theoreticians in their turn not always consider it necessary to become familiar with the work of working teachers. Meanwhile, comparisons and contrasts between different approaches are evidently fruitful.

The composition of textbooks is undoubtedly based on definite theoretical principles, even if these are not always explicitly formulated, and perhaps not even fully recognized by the authors themselves. Analyzing existing experience and identifying such principles, we would argue, is useful to working teachers and theoreticians alike.

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