TYPOLOGIES OF MATHEMATICAL PROBLEMS: FROM CLASSROOM EXPERIENCE TO PEDAGOGICAL CONCEPTS

Oleg A. Ivanov, Tatiana Yu. Ivanova, and Konstantin M. Stolbov

St. Petersburg State University, Lyceum For Physics and Mathematics # 30, Lyceum "Physical-Technical High School", St. Petersburg, Russia

In this paper we propose a new classification of mathematical problems. Usually one speaks about low-level and high-level problems, about exercises and authentic mathematical tasks. We are going to classify problems in terms of activities needed to solve them and schemes of solutions. Moreover, a two-dimensional model of classification is introduced. Finally, we suggest an approach aimed at the development of the inductively-associative form of thinking.

Keywords: classification of problems, schemes of solutions, kinds of activities, associative thinking

INTRODUCTION

One of the main problems of teaching mathematics is the selection of tasks which 'promote and develop mathematical talent' in the best way (see, for example, Chamberlin, 2010). V. G. Dorofeev suggested using tasks with rich problem neighborhood (Dorofeev, 1983). The idea of the recent article of R. Leikin (Leikin, 2007) was to teach students to find diverse solutions of a given problem, which form what the author calls the 'solution space of a mathematical problem'. One of the authors of this article introduced a notion of 'a problem cluster', that is, 'a collection of problems possessing a complex inner structure' (Ivanov, 1998, and Ivanov & Il'ina, 2001). On the other hand, since problem solving is an *activity*, may be, it would be reasonable to distinguish problems by activities and methods that are used while solving them.

THE STARTING POINT

First of all, we are going to present a set consisting of ten problems. In our opinion, the top priority of any educational process must be the mathematical (intellectual) development. Studying mathematics, pupils solve a lot of problems. However, problem solving means more than just a straightforward execution of standard procedures. We must bear in mind that a famous Soviet psychologist S. L. Rubinstein said that: "Man's development does not coincide with the content of his knowledge and skills and is not determined by the coherence of operations inherited to man, but by the culture of his internal intellectual processes." Basing on our experience and intuition, we tried to select problems by means of which we could, so to speak, "enter the child's mind". Of course, these problems should be as varied as possible.

Our experience says that these problems are (more or less) accessible to pupils aged 16-17 studying mathematics at the advanced level at the St. Petersburg Lyceum # 30 for Physics and Mathematics. However, unfortunately, they appear to be too difficult for the vast majority of Russian pupils (of the same age). We agree that the statements of these problems may appear to be rather unexpected, but there is no "complicated mathematics" in these problems, no one needs to know "sophisticated techniques" to solve them, and all necessary notions and methods are well-known to all pupils. Thus, if they have a very simple structure, what makes them so difficult?

Our investigation started when we posed the following question to ourselves:

• In what terms can one describe the characteristics of these problems in order to use them in the construction of the learning process and, in particular, in a classroom practice?

In a certain sense we looked for a transition from 'the tacit expertise' to 'the grounded science' (Ruthven, 1993). The problem is that it is impossible to describe these characteristics in terms of the typologies of mathematical problems that are known in Russia and other countries (see, for example, Sarantsev & Miganova, 2001, and Chamberlin, 2010). For example, Chamberlin (Chamberlin, 2010) says that "HOT (*high-level*) tasks are those in which the problem solver needs to engage in cognition to successfully solve the problem." But what if a pupil is able to solve one HOT task but isn't able to solve another? Can the cognition processes in these cases differ from each other?

We are going to introduce a new typology (in a certain sense a two-dimensional one). Some of its components were known previously (Vedernikova & Ivanov, 2002). To justify this typology, we are going to present and discuss the solutions of the following problems.

A SET OF PROBLEMS

1. The numerator of a given fraction is increased by 1 and the denominator is increased by 2. Compare the fraction obtained with the given fraction.

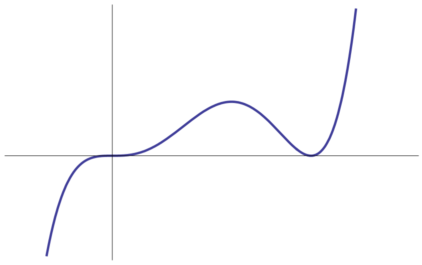
2. Find the range of the expression $a^2 - 2a + b$ for $a \in [-2,3]$ and $b \in [-2, 1]$.

3. How many real solutions does the equation $2^x + 2^y = 2^{x+y}$ have?

4. Check whether or not the number 100903027 is prime.

5. Suppose the numbers x^{199} and x^{213} are both rational. Is it true that the number x is also rational?

6. Give a formula for a function whose graph looks like a curve in the following figure.



7. Find the largest value of the fraction $\frac{n^2}{2^n}$ where n is a positive integer.

8. Does there exist a line tangent to the parabola $y = x^2 - x + 5$ and parallel to the line $y = 2011x$?

9. Let a, b, and c be the sides of a triangle and S be its area. Prove that $6S < ab + bc + ca$.

10. Consider the systems of the form

$$\begin{cases} x * 1 \\ x * 2 \\ x * 3 \\ x * 4 \end{cases}$$

where a "*" stands for one of the symbols ≤ or ≥. Find the number of systems having nonempty solution set.

FORMAL SOLUTIONS OF THE PROBLEMS

In this section, we present solutions of these problems (which are given by the authors). Pupils' solutions (and mistakes) will be discussed further.

Problem 1. Let k be the numerator of the given fraction and let n be its denominator. After increasing the numerator by 1 and increasing the denominator by 2 we obtain the fraction $\frac{k+1}{n+2}$. In order to understand which fraction is greater, let us consider their difference. Thus we obtain $\frac{k}{n} - \frac{k+1}{n+2} = \frac{2k-n}{n(n+2)}$. Consequently, the initial number is greater if $2k > n$, or $\frac{k}{n} > \frac{1}{2}$. Thereby, if the given number was greater than 0.5, then we'll obtain the number which is less than the given one. And if the given number was less than 0.5, then the obtained number is greater than the given number.

Problem 2. Let us rewrite the given expression in the form $(a+1)^2 - b - 1$. Since $-1 \le a + 1 \le 4$, the interval [0, 16] is the range of the expression $(a+1)^2$. Since $-2 \le b \le 1$, the interval [−2, 1] is the range of the expression $-b - 1$. Consequently, the interval [−2, 17] is the range of the given expression.

Problem 3. Let us express y in terms of x by a sequel of natural transformations,

$$2^{x+y} - 2^y = 2^x, 2^y(2^x - 1) = 2^x, 2^y = \frac{2^x}{2^x - 1},$$

so $y = \log_2 \frac{2^x}{2^x - 1} = x - \log_2(2^x - 1)$. Thus, for any $x > 0$, there exists a unique y such that the pair (x, y) satisfies the given equation. This equation has infinitely many solutions.

Problem 4. The given number can be rewritten in the form

100900000+3027 = 1009 · 100000 +3 · 1009 = 1009 · (100000 +3) = 1009 · 100003.

Therefore, it is a composite number.

Problem 5. Since the numbers x^{213} and x^{199} are rational, their quotient $\frac{x^{213}}{x^{199}} = x^{213-199} = x^{14}$ is rational, too. Similarly, the quotient of the number x^{199} and the fourteenth power of the number x^{14} is rational, so the number $x^{199-14\cdot14} = x^3$ is rational. Moreover, the number $\frac{x^{14}}{x^{12}} = x^2$ is rational; consequently, the given number $x = \frac{x^3}{x^2}$ is also rational.

Problem 6. Let us try to find a polynomial $p(x)$ whose graph looks like the given curve. Number $x = 0$ must be its root with multiplicity three or more and the number $x = 2$ must be a root with multiplicity two or more.

Let $p(x) = x^3(x-2)^2$. We have to check that the behavior of the function is similar to a function with a given graph, in particular, that the function $p(x)$ has three intervals of monotonicity. For that purpose we find the derivative,

$p'(x) = 3x^2(x-2)^2 + 2x^3(x-2) = x^2(x-2)(5x-6)$.

Therefore the function $p(x)$ increases on the intervals $(-\infty, 6/5]$ and $[2, +\infty)$ and decreases on the interval $[6/5\,, 2]$.

Problem 7. Let us solve the inequality $\frac{n^2}{2^n} < \frac{(n+1)^2}{2^{n+1}}$. Multiplying both sides by the common denominator we obtain the inequalities $2n^2 < n^2 + 2n + 1$, $n^2 < 2n + 1$, and $(n-1)^2 < 2$, which are valid only for $n = 1{,}2$. Consequently, the fraction attains its largest value for $n = 3$. Thus, the answer is 9/8.

Problem 8. The slope of the line tangent to the parabola $y = x^2 - x + 5$ at the point (x_0, y_0) equals $2x_0 - 1$. This tangent line is parallel to the line $y = 2011x$ if and only if $2x_0 - 1 = 2011$, that is, if $x_0 = 1006$.

Perhaps, it would be more natural to use the condition of tangency of a line to a parabola in terms of multiplicity of roots of an equation. Indeed, the line $y = 2011x + b$ is tangent to the graph $y = x^2 - x + 5$ if and only if the equation $x^2 - x + 5 = 2011x + b$ has a unique solution, that is, if the discriminant of the quadratic equation $x^2 - 2012x - (b + 5)$ equals zero. Obviously, the equation $2012^2 + 4b + 20 = 0$ has a solution; thus, the desired tangent line exists.

Problem 9. The inequality $2S \le ab$ is valid; for example, since $2S = ab \sin\alpha$, where α is the angle between the sides of the length a and b of a triangle. The equality takes place if and only if α= 90˚. Therefore, $6S \leq ab + bc + ac$. Now, since in a triangle, there can be only one right angle, at least two among three inequalities $2S \leq ab$, $2S \leq bc$, and $2S \leq ac$ are strict ones, so $6S < ab + bc + ac$.

Problem 10. Let us consider one of those systems. Since the solution set of each of the inequalities constituting this system is an interval and since the intersection of intervals is also an interval, the solution set of this system coincides with an interval. The endpoints of this interval lie in the finite set {1, 2, 3, 4, 5}, whence it coincides with one of the intervals $(-\infty, 1]$, $[1, 2]$, $[2, 3]$, $[3, 4]$, and $[4, +\infty)$. Therefore, there are five systems with a nonempty solution set.

SCHEMES OF SOLUTIONS

We are sure that the development of mathematical thinking has to be one of the main goals of teaching mathematics at schools. First of all, pupils must see and perceive that it is a reasoning that constitutes the core of the solution of any mathematical problem. Now, if we want pupils to be able to write down a rigorous solution, he or she has to understand what is a justification (a proof) and what is not. The majority of mistakes arose from pupils' inability to think in a logically-deductive way.

The mistakes that pupils made in their solutions of Problem 1 are the typical examples. Some of them wrote that "since the denominator of the fraction increased by a bigger number than its numerator, the value of the fraction decreased" without giving any justification of this assertion. Here is the typical example of the solution of Problem 5. "Since the power of an irrational number may be a rational number, the rationality of x^{199} and x^{213} does not imply the number x is rational." Pupils did not manage to see the gaps in their reasoning and, as a result, they did not even try to find the correct solutions.

Let us now examine solutions of problems 3, 5, 6, and 9 from the point of view of the kinds of ways of reasoning. The scheme of the solution of Problem 3 could be called algebraic. One has to only use the well known algebraic relation $2^{x+y} = 2^x \cdot 2^y$ to obtain a linear-fractional equation in 2^x and 2^y. The construction of an algorithm is the key for the solution of Problem 5. We call such a scheme combinatory-algorithmic. Although an answer to Problem 6 is a polynomial, it follows from analyzing properties of a function. The corresponding scheme may be called analytical. Finally, there are no transformations or algorithms in the solution of Problem 9, only pure logic. Certainly, any solution is an example of a logical reasoning; thus, we suggest calling such schemes syllogistic (in particular, a proof built upon reductio ad absurdum is syllogistic).

Thus, we suggest classifying problems according to schemes involved in their solutions: *algebraic*, *analytical*, *combinatory-algorithmic*, *syllogistic*.

Obviously, though in the solution of a really hard mathematical problem various kinds of schemes of reasoning occur, it seems natural to select problems requiring the above-mentioned schemes in their solutions (in their purest form, so to speak).

KINDS OF (MATHEMATICAL) ACTIVITIES

Now let us look at the solution of problem 7. It is short and simple, but how can one find it? A usual situation in a classroom is the following one: pupils look at the statement of the problem, sit still, and do not know what to do. The solution of any problem is a result of a (mathematical) activity. Unfortunately, Russian pupils are still used to only solving problems by means of prescribed rules. Usually, they have to only recognize the problem in order to use the corresponding method. As a result of such teaching and learning, they get stumped if more than one method is needed to solve a problem. However, let us suggest to pupils to calculate the initial terms of the given sequence $x_n = \frac{n^2}{2^n}$. The following table contains the result of the calculation.

n	1	2	3	4	5	6
x_n	0.5	1	1.125	1	$\frac{25}{32} \approx 0.781$	$\frac{9}{16} \approx 0.563$

Now the behavior of the given sequence becomes completely clear. One only has to prove that this sequence decreases starting with its third term. Thus, we carried out a mathematical experiment that resulted in setting a hypothesis. The proof of this hypothesis was given in the above solution.

The mathematical experiment that could be carried out in a solution of Problem 1 shows that the obtained fraction may be more or less than the given one. Indeed, $\frac{1}{3} < \frac{2}{5}$, though $\frac{3}{2} > 1 = \frac{4}{4}$. Although we aren't able to set the correct hypothesis, this experiment will save us from the wrong conclusion. The solution itself consists of three steps. First of all, we have to codify the statement of the problem in order to write it in a symbolic form, namely, the fraction $\frac{k}{n}$ is given and the fraction $\frac{k+1}{n+2}$ is obtained. On the second step we rewrote the inequality $a > b$ in the form $a - b > 0$ and transformed the difference of fractions. Finally, we have to interpret correctly the obtained condition on the numbers k and n. Indeed, if we rewrite the inequality $2k > n$ in the form $\frac{k}{n} > \frac{1}{2}$, we obtain the condition on the value of the given fraction.

What kind of activities occurred in searching for the solution of Problem 4? As for us, we call it restructuring. Perhaps, this form of activity appears more transparent in the solution of the following problem.

Problem 11. Check whether or not for any positive integer n the number $n^5 + n^3 + n^2 + 1$ is a composite.

The point is that when one is factorizing polynomials he or she is restructuring an expression. In our case, since $n^5 + n^3 + n^2 + 1 = n^3(n^2 + 1) + n^2 + 1 = (n^2 + 1)(n^3 + 1)$, this number is a composite for any positive integer n. One is engaged in the same kind of activity doing substitutions in order to reduce a transcendental equation to an algebraic one, or, more generally, representing some given function as composition of other simpler functions. To conclude our discussion of Problem 4, we want to stress that the mathematical experiment will fail since the smallest divisor (over than 1) of the given number equals 1009, which is the 169th prime number.

We suggest considering such kinds of general activities as: *using a standard method, mathematical experiment, codifying, interpreting, restructuring, setting a hypotheses, self-control, transforming and transferring*.

Self-control is needed to avoid mistakes. The first example of using self-control one could see in the solution of Problem 1. Indeed, on one hand, almost everyone knows that $2^{x+y} \neq 2^x + 2^y$. On the other hand, this "non-equality" means that the relation $2^{x+y} = 2^x + 2^y$ cannot be valid for all pairs $x, y \in R$, though almost everyone could see that $2^1 + 2^1 = 4 = 2^{1+1}$. Therefore, the pair $(x, y) = (1,1)$ is the solution of the given equation. It is impossible to guess other solutions; however, this does not imply that they do not exist! It may seem strange that we consider self-control as *an activity*. The point is that a skilled mathematician is able to catch his or her own mistakes unconsciously, seeing that the obtained result could not be valid. By contrast, a novice learner quite often ignores the obvious absurdities. Because of this a teacher has to foster self-control abilities which can be achieved only as a result of purposeful activity.

By transforming and transferring we mean, in particular, using ideas and methods from some themes of mathematics to solve problems initially stated in the terms from another theme. For example, in the second solution of problem 8 we transformed the problem from calculus to the problem concerning multiple roots of polynomials. The same kind of transformation occurred in the solution of Problem 6.

A TWO-DIMENSIONAL MODEL

The main idea consists of using the classification "kind of activity—scheme of reasoning" to assess the process of mathematical education. If we want to develop different aspects of students' mathematical thinking such as ability to formalize mathematical material, to generalize it, and others (Krutetskii, 1976), given problems have to be diverse in terms of kinds of activity used for their solution, and in terms of schemes of reasoning. In the following table we present our analysis of the solutions of the discussed problems.

#	kind of activity	scheme of reasoning
1	codifying; interpreting	algebraic
2	restructuring	analytical
3	self-control	algebraic
4	restructuring	algebraic
5	self-control	combinatory-algorithmic
6	transforming and transferring	analytical

7	experiment; setting a hypotheses	analytical
8_1	using a standard method	analytical
8_2	transforming and transferring	algebraic
9	using a standard method	syllogistic
10	experiment	combinatory-algorithmic

As you could see, there are no problems with coinciding pedagogical characteristics; all these problems are distinct. Therefore, in particular, this set may be used successfully by a teacher for assessment of pupils' mathematical development.

PAIRING PROBLEMS

The problem of the development of inductively-associative form of mathematical thinking is highly important. Obviously, it is harder to attain this goal than to teach pupils to reason logically and rigorously. In the last part of our paper, we are going to develop the approach suggested in the section "Instead of a Conclusion" of the book Ivanov, 2009. The main (and well-known) idea is that "What you have been obliged to discover for yourself leaves a path in your mind which you can use again when the need arises." George Christoph Lichtenberg (1742-1799).

In order to foster the associative thinking it is important to accustom pupils to see interrelations between various problems that may seem to be distinct. We restrict ourselves with several examples. We'll accompany each of these examples with a "name" (the theoretical understanding of the relationships manifested is a task for the future).

Example 1 ("Push on the idea"). In a situation when your pupils aren't able to solve the suggested problem, it is worth giving them another one. For example, you gave them Problem 4. It is likely that they cannot solve it.

You may suggest Problem 11 by saying something like "Never mind, try to solve another problem". It is likely that the majority of your pupils will do it. Your next sentence should be: "Surely you have not guessed how to solve the previous problem?"

Example 2 ("The mathematical content"). Consider the following problem.

Problem 12. Examine the behavior of the function $f(x) = \frac{x+a}{2x+b}$ (here a and b are positive numbers) on the interval [0,+∞).

Its solution is rather standard. For example, one can find the derivative, $f'(x) = \frac{2x+b-2(x+a)}{(2x+b)^2} = \frac{b-2a}{(2x+b)^2}$. Thus, if $b > 2a$, then $f'(x) > 0$, which implies that this function increases on the given interval; similarly, if $b < 2a$, then $f'(x) < 0$ and the function is decreasing. It is worth noting that the condition $b > 2a$ is equivalent to the condition $f(0) = \frac{a}{b} < \frac{1}{2}$.

Now let us pose two questions.

Question 1: *"Can you justify without doing any calculations that the inequality* $f(0) > \frac{1}{2}$ *implies that the given function decreases on the interval* [0,+∞)*?"*

Question 2: *"Can you see some relationship between this problem and a problem within the given set?"*

The point is that Problem 1 is a straightforward corollary of Problem 12. And the number 0.5 in the answers of these problems appears because of the fact that the line $y = 0.5$ is the asymptote of a hyperbola which is the graph of the given linear-fractional function.

By the way, do you agree that it is useful to pair problems?

One more example of disclosure of the mathematical content is another solution of Problem 5. Any mathematician would reason in the following way. Since numbers 213 and 199 are relatively prime, there exist integers a and b, such that $213a + 199b = 1$, hence the number $x = x^{213a+199b} = (x^{213})^a \cdot (x^{199})^b$ is rational. One could check that the two given numbers are relatively prime using the Euclidean algorithm. And this is exactly what has been done in the above solution.

Example 3 ("The relationship with a known method"). There is a well-known (even routine) method of solving the following problem.

Problem 13. Solve the inequality $(x-1)(x-2)(x-3)(x-4) \geq 0$.

Certainly, one can reason in the following way. The product of four numbers is positive if all these numbers are positive or within them there are two or four negative numbers. Thus, one can solve the given inequality by examining eight systems of inequalities. However, there exists much shorter solution. The points 1, 2, 3, and 4 divide the line into five intervals. The left-hand side of the inequality does not change its sign within any of these intervals. Thus, the solution coincides with the union of the whole intervals and what we only have to do is to determine the sign in each interval. Certainly, it is the discussed method that should have appeared in pupils'mind in connection with Problem 10.

CONCLUSION

We can't judge about other countries, but in Russia teaching mathematics even at the advanced levels too often means widening of studied mathematical notions, methods and algorithms. But, as it was mentioned at the conference "Teaching mathematics in Mathematics and Science high schools" (Saint-Petersburg, 2012), we can't attain deep understanding of mathematics only by suggesting to students a lot of problems, most of which in our textbooks (including the textbooks for mathematics schools) are the problems about applying known methods and algorithms. As a result, the majority of our pupils are only able to apply standard methods and become helpless even in the simple situation when they have to use two techniques in order to solve a problem. We hope that the results of our research will be useful for teachers, supplying them with instruments which help them to analyse and enrich their pedagogical strategy as well as help them in constructing concrete lessons.

To conclude, the proposed ideas may be also instructive in pre-service and in-service mathematics teacher education (see, Ivanov & Il'ina, 2001, Ivanov, 2001, and Ivanov, 2009).

REFERENCES

Chamberlin, S. A. (2010). Mathematical Problems That Optimize Learning for Academically Advanced Students in Grades K–6. *Journal of Advanced Academics, 1*, 52–76.

Dorofeev, G. V. (1983). O sostavlenii tsiklov vzaimosvyazannykh zadach (On the designing cycles of interconnected problems). *Mathematics at School*, *6*, 34–39. (In Russian)

Ivanov, O. A. (1998). Problem Cluster Within a Curriculum: An Integrated Approach to Instruction. In *Proceedings of CIEAEM–49*, Setubal, 286–292.

Ivanov, O. A. (1999). *Easy as π? An Introduction to Higher Mathematics*, Springer–Verlag New York, Inc.

Ivanov, O. A. & Il'ina, A. N. (2001). Problem Solving in and Understanding of Mathematics in Teacher Education. *Der Mathematik–Unterricht, 6*, 34–45. (In German)

Ivanov, O. A. (2001). Praktikum po elementarnoj matematike. Algebro-analiticheskie metody (*Workbook on high school mathematics: An algebra– analytical approach*), MCCME, Moscow. (In Russian)

Ivanov, O. A. (2009). *Making Mathematics Come to Life: A Guide for Teachers and Students*, American Mathematical Society, Providence, RI.

Krutetskii. V. A. (1976). The psychology of mathematical abilities in schoolchildren. (J. Teller, Trans., J. KIlpatric & I. Wirszup, Eds.). Chicago: The University of Chicago Press.

Leikin, R. (2007). Habits of mind associated with advanced mathematical thinking and solution spaces of mathematical tasks. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the Fifth Conference of the European Society for Research in Mathematics Education — CERME-5*, 2330–2339.

Ruthven, K. (1993). Pedagogical knowledge and training of mathematics teachers. *Mathematics Education Review, 3*, 1–10.

Sarantsev, G.I. & Miganova E.Yu. (2001). Funktsii zadach v protsesse obucheniya (Functions of problems in the learning process), *Pedagogika, 9*, 19-24. (In Russian)

Vedernikova, T. N. & Ivanov, O. A. (2002). Intellektual'noe razvitie shkol'nikov na urokah matematiki (Pupils' intellectual development on mathematics lessons), *Mathematics at School, 3*, 41–45. (In Russian)