

STUDENTS' EMERGING MODELS OF AVERAGE RATES OF CHANGE IN CONTEXT

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This paper describes a model application activity in the context of how light intensity varies depending on the distance from a point light source within a model development sequence focusing on developing students' understanding of average rate of change. Students' early models were found to be linearly decreasing functions and they did not distinguish between the light intensity and changes in light intensity. Most students were able to draw on several representations (most notably tables and graphs) to represent, describe and distinguish between the light intensity and changes in light intensity. Some students encountered difficulties when describing change when the rates were negative and understanding how the graphical representation of average rates of change depended on the interval width.

The difficulties encountered by students when reasoning about and interpreting rates of change are well documented in the research literature (Carlson et al., 2002; Monk, 1992; Thompson, 1994). The covariational reasoning students need to simultaneously attend to both the changing values of the outputs of a function and the changing values of the inputs to the function, is foundational for understanding average rates of change in pre-calculus and instantaneous rates of change in calculus (Johnson, 2012; Oehrtman, Carlson & Thompson, 2008). However, relatively little research has attended to the particular challenges students encounter when reasoning about negative rates of change, the role of context in the tasks involving change, and students' construction and application of representations. Many studies investigating aspects of average rates of change use different representations (graphs, tables, figures, symbols, written language, simulations, enactments) to create tasks on which students' work is then analyzed (e.g. Johnson, 2012; Herbert & Pierce, 2012). Often, the use of different contexts provides a background situating the representations, and students are asked to extract contextual meaning from a given representation.

Whitney (2010) investigated students' initial understandings of rate of change and their preference for using different representations (tables, graphs, equations and contexts) in the context of linear functions. The students tended to use tables as tools for organizing information and "graphs were not students' first choice in representing information" (p. 186). Along the same line, Herbert and Pierce (2012) found that the rate-related information understood by students through different representations was highly individual, and that "understandings of rate in one representation or context are not necessarily transferred to another representation or context" (p. 455). This is consistent with the findings of Ibrahim and Rebello (2012) comparing student work with different representations in the subject matter areas of kinematics and work.

In this study, we examined students' use of average rates of change to construct, interpret and describe representations of the relationship between measurements of light intensity and distances from a light source. We focused our attention on students' descriptions of how the light intensity varies with the distance from the light source and on the students' interpretations of the graphs that they created to represent the average rates of change. The questions guiding this investigation were: How do students interpret and describe their constructed representations of average rates of change? How do students attend to context when interpreting and describing their constructed representations of average rates of change?

THEORETICAL BACKGROUND

We used what Kaiser and Sriraman (2006) call a contextual modeling perspective on teaching and learning mathematics (cf. Lesh & Doerr, 2003) and a design-based research methodology (Cobb et al 2002; Kelly et al. 2008) to design and implement a sequence of modeling tasks aimed at supporting the development of students' understandings of average rates of change.

Model development sequences

Model development sequences (Doerr & English, 2003; Lesh et al., 2003) begin with model eliciting activities where students are confronted with the need to develop a model in order to make sense of a situation familiar to the students. When the model is elicited, the purpose of successive model exploration activities and/or model application activities is to help the students extend, revise and refine their emerging models. The focus of model exploration activities is on the underlying structure of the elicited model and especially on the strengths of various representations as well as on how to productively use these different representations. In model application activities the students apply their model to new situations, which in turn can lead to the model being further extended, revised and refined and to further understandings of the utility of the representations.

METHODOLOGY

The model development sequence designed to support the development of students' understandings of average rates of change formed the basis for a six-week course taught by the third author for students who were preparing to enter their university studies. Throughout this sequence, students were engaged in the processes of iteratively revising and improving their models of average rates of change while interacting with other students and participating in teacher-led class discussions. A total of 34 subjects (10 female and 24 male) participated in the study; 21 students had studied calculus in high school and 13 had not studied any calculus. The students worked in small groups to complete the different activities. The model eliciting activity used motion detectors to elicit the notion of negative velocity with motion along a straight path. To explore changing negative rates, multiple model exploration activities were used based on computer simulations (Kaput & Roschelle, 1996) and structured exercises (Watson & Mason, 2006) focusing on different representations of

negative rates. Two model application activities in the sequence were carried out as lab projects where the students spent several days on the activities and wrote final reports. One model application activity focused on creating a model of the intensity of light with respect to the distance from a light source (the Light Lab), and the other aimed at modeling the rate at which a fully charged capacitor in a resistor-capacitor circuit discharged with respect to time. Each activity included class discussions focusing on the relationships among different representations of negative rates of change and students' interpretation and descriptions of change in different contexts and settings. The work reported on in this paper is the analysis of the students' work on the Light Lab, which took place at the beginning of the fourth week of the course.

The Model Application Activity Design

The Light Lab task was designed and carried out in six phases as illustrated in Figure 1. Prior to the Light Lab, the students' work on the model eliciting and exploration activities almost exclusively involved different representations of one-dimensional motion along a straight path. The aim of the Light Lab was manifold; it was originally designed as a model application activity in that it provided the students with an opportunity to apply their emerging models of average rate of change elicited in the context of linear motion in a new context. However, the iterative design process inherent in the design-based methodology led to the successive inclusion of elements of exploration focusing on the graphical representations of the average rate of change in non-linear contexts.



Figure 1. The six phases of the Light Lab

The first pre-lab task aimed at making explicit the students' intuitive and initial models about the relationship between the intensity measured at different distances from a point source. The students were asked to consider a one-dimensional scenario of an approaching car and to sketch a graph of how the intensity of the car's headlights varied depending on the distance you are from the car. The students were also asked to describe how light is emitted from a point source in terms of how the light rays emitted from it are dispersed, and what implications this might have for their emerging intensity model. The students' work on this task was collected and discussed in class.

In the second task, students collected light intensity data using a flashlight with the focusing cap removed, a meter stick, and a light sensor connected to their graphing calculator. They collected 15 measurements of the intensity at one cm intervals from the light source and transferred the data to their computers. The light sensor measured light intensity in a relative and arbitrary unit called "light intensity units" (LIU). Due

to physical limitations of the light sensor, the first two or three data points gave a maximum reading of one LIU on the light sensor.

In part one of the lab, the students were asked to make a scatter plot of their data and to write descriptions of how the intensity of the light changed with respect to distance from the light source. They were encouraged to compare this relationship to their predictions from the first pre-lab. The students were also asked to calculate the average rates of change of the data in one cm intervals and to (1) describe the average rate of change of the data as distance from the light source increases; (2) to graph the average rates of change in a scatter plot; and (3) to create a separate rate graph of the calculated average rates of change.

In the second pre-lab task, the students were given four images representing light intensity at different distances. The number of dots per square inch indicated the intensity, and the task for the students was to determine the intensity in dots per square inch at a given distance from the light source. The second pre-lab introduced an inverse square relationship as a model for how the intensity varies with distance from the light source, and familiarized the students with finding such a functional expression fitting a given data set. This task was given as homework and subsequently discussed in class.

In the second part of the lab, the students were asked to determine a function fitting their collected data, explain their work, and analyze the average rates of change of their function using the difference quotient. Due to the limitations of the sensor, students needed to construct a piecewise function, consisting of a horizontal linear piece for the first two or three data points and an inverse squared piece for the remainder of the data set. More specifically, the student were asked (1) to calculate the average rates of change of the function at integer x values and an interval width of 0.5 cm; (2) to show these average rates of change in a function plot; (3) to create a rate graph for the average rates of change values; and (4) to describe how the average rate of change of the function changes as distance from the light source increase.

Data analysis

In this paper, we report on the analysis of the student data from pre-lab 1 ($n=34$; done individually) and the lab reports ($n=18$; all written by pairs of students with the exception of one student writing alone). The analysis was done in two phases following the principles of grounded theory (Strauss & Corbin, 1998). First, codes were developed to categorize the students' reasoning and answers on each of the questions for pre-lab 1. This initial analysis focused on capturing the students' initial models of how the light intensity varies with distance from the light source and how light disperses from a point source. In the second phase of the analysis, the students' final lab reports were read and coded, focusing on interpretations and descriptions of how the intensity varied with the distance from the light source and its average rates of change, the graphs the students produced, how they attended to the context throughout their writing, and how the students' used their ideas from the pre-labs.

RESULTS

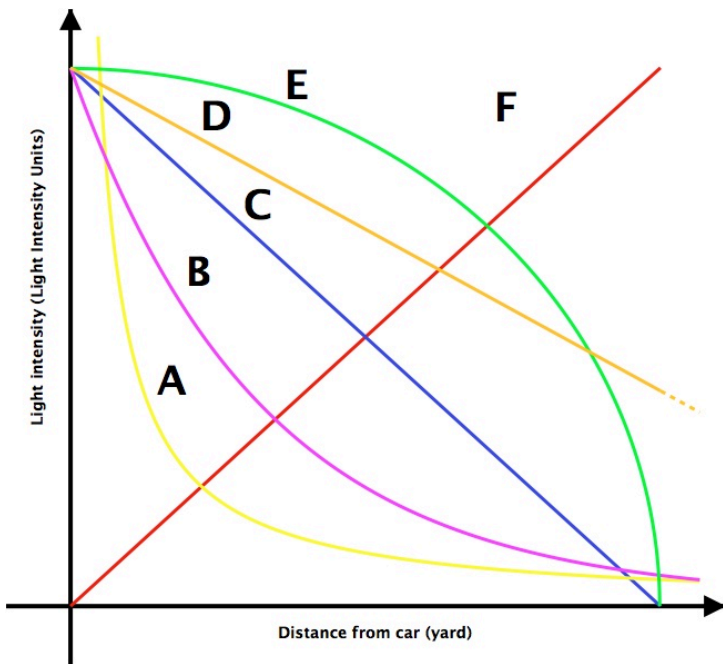
Our analysis yielded three sets of results. First, we found that most students' early model of changing light intensity was a linearly decreasing function and did not distinguish between the light intensity and changes in light intensity. Second, as students' models of light intensity with respect to distance developed, most students were able to draw on several representations (most notably tables and graphs) to represent, describe and distinguish between the light intensity and changes in light intensity. Third, some students encountered difficulties when describing change when the rates are negative and understanding how the graphical representation of average rates of change depends on the interval width.

Students' initial models of variability of light intensity at different distances

The students' initial models of the relationship between light intensity and the distance from the light source are shown in Figure 3, which illustrates the different types of graphs the students constructed to describe how the light intensity from a car's headlights varies with the distance to the car. The table displays the number of students' graphs in each category; the cut off between the C and the D graph was a slope of -0.7 . Nearly all of the students ($n=28$, 83%) drew a linear relationship between the intensity of light and the distance from the source. All but one of these linear graphs (shown as C and D in Figure 3) correctly show the intensity decreasing, but incorrectly at a constant rate. This is likely due to the students assuming that the speed of the approaching car is constant (c.f., Doerr, Arleback & O'Neil, submitted) and confusing the constancy of speed of the car with the constancy of the decrease in light intensity. The one student who drew graph F incorrectly reasoned that the light intensity increased as the distance from the car increased. The four students who drew graph A, with its asymptotic behavior at the y-axis, were likely drawing on their formal physics and/or mathematical knowledge of the inverse square law for light intensity. We note that all students had taken a prior course in physics. Since graph B shows a maximum intensity at the instant the distance is zero, the one student who drew graph B may have attended to his perception of the context as well as his formal previous knowledge.

Nearly all students ($n=26$, 76%) provided a qualitative description of the intensity at 1000 yards compared to at 2000 yards, using vocabulary like "greater" and "less". The remaining 24% of the students, on the other hand, made a quantitative statement describing the intensity at 1000 yards to be exactly twice the intensity at 2000 yards. This was done exclusively by students drawing a C graph in Figure 3. While it is possible that this could be the case, this reasoning is more suggestive of exponential decay (whose change is multiplicative in structure) than a linear function, whose change is additive in structure.

When asked to "Compare the rate at which the intensity is changing at 1000 yards and 2000 yards", 17 of the 34 students described or calculated the rate of change their graphical representation. One student, referring to his C graph, wrote: "The rate at



Graph	Number of students
A	4 (12%)
B	1 (3%)
C	22 (65%)
D	5 (15%)
E	1 (3%)
F	1 (3%)

Figure 3. Students’ initial models of intensity vs. distance from light source

which the intensity is changing at 1000 yards and 2000 yards is the same”. However, 16 of the 34 students compared the values of the function at the two distances rather than the rate of change at those distances. For some of these students, their description did not mention the rate of change at all (for the C graph: “The intensity doubles from 2000 yards to 1000 yards”); other students explicitly confounded the function’s values with the rate of change values (C graph: “The rate of intensity is doubled”). The remaining one of the 34 students simply gave an equation of a linear function that did not correspond to his graph.

When asked to “Draw some representative light rays leaving the light source” the students drew figures of light dispersing either as cones, perpendicular rays or waves

A. Light rays “cone-like”	B. Light rays perpendicular	C. Light disperse as waves
28 students (82%)	5 students (15%)	1 student (3%)

Figure 4. Students’ initial models of intensity vs. distance from light source

(see Figure 4). Students drawing a cone-like model have a potential rationale for why the light intensity decreases when distance increases, whereas the perpendicular model implies a constant light intensity, independent of distance. One student, likely

drawing on previous knowledge from prior coursework in physics, drew the light dispersing as waves. However, regardless of their model in Figure 4, all students concluded that the light intensity would decrease as distance increase. This illustrates the compartmentalization of conflicting ideas in the students' emerging models that the students have still not resolved and pulled together.

Students' descriptions of the light intensity and its average rate of change

All students used tables to organize and summarize their raw data and various calculated quantities. Scatter plots of the data were included in all reports, either as separate graphs, together with graphs where the average rates of changes were drawn in, or, in graphs showing how their symbolic piece-wise function fit the data. Nearly all (89%) of the reports included correct descriptions of the values of the data drawing on the context of light intensity. Some descriptions tended to be short and close to strict mathematical statements: "The intensity of the light source decreases as the distance increases." Other descriptions were more closely situated and connected to the lab setting: "As the distance from the light source increased, the light intensity decreased. The farther away the sensor was away from the light sensor the smaller the light intensity became." These two sentences convey the same information using slightly different language: the first is a general sentence about the behavior of how light intensity varies with distance from the light source, similar to the more mathematical description in the previous example. This relationship is then expressed in terms of the context of the lab. One can also note the use of the word "smaller" in the last sentence, which does not quite capture the notion of light intensity, pointing at the complexity of trying to make precise and meaningful interpretations of mathematical statements and relationships in context.

In some cases, the students made explicit references back to their data to support their descriptions. In the following example, the description accompanied a table listing the light intensity values and the calculated average rates of change values as well as a scatter plot of the data with the average rates of change drawn in, connecting all consecutive points (as shown in Figure 5A): "As the distance from the source increases the intensity of the light decreases. For example, at 3 cm the light intensity reads .832 LIU [Light Intensity Units]; at 4 cm the light intensity reads .637 LIU; and at 5 cm the light intensity reads .463 LIU. It is also noted that rate at which the data changes as the distance from the light source increases changes." This description also includes a correct conclusion about the long run behavior of the average rates of change of the light intensity. Over half (56%) of the students constructed and correctly interpreted a similar graph of the average rates of change. In this particular case, the students continued to elaborate their description of the behavior of the average rates of change: "Although the majority of the average rates of change are increasing, the absolute value of average rates of change are decreasing which tells us that as the distance from the light source increases the intensity of the light decreases quickly at first and then decreases more slowly." In this statement, the students argued from the graph of the average rates of change values (Figure 5B), and

articulated that these values are increasing. But the students then considered the absolute values of the negative average rates of change to provide the qualitative statement about the rate that is “decreasing quickly at first and then decreases more slowly.” All reports but two (94%) included the correct rate graphs of the average rate of change over the 1 cm intervals similar to the one in Figure 5B and 44% of the student made it explicitly clear how to see the average rates of change in the scatter plot (see Figure 5A) by referring to the slope of the line segment connecting two consecutive points.

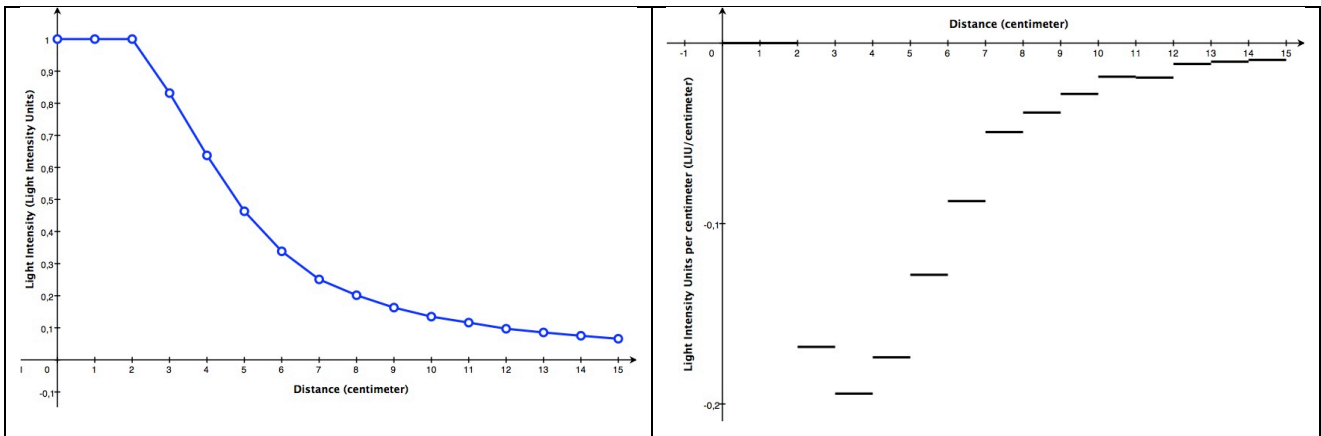


Figure 5. (A) shows the scatter plot and (B) the corresponding rate graph of typical data the students collected

Students’ difficulties describing and representing change

For some students, their descriptions of the behavior of the light intensity contradicted their description of the average rate of change of the light intensity. For example: “The intensity of the light decreases at a decreasing rate with respect to the distance from the light source, as predicted earlier. The average rates of change increase at a decreasing rate as distance from the light source increases.” This points to the complexity of the ideas involved in reasoning with negative rates and their different representations, especially when cast in an applied context.

Some of the mathematical demands of the model application activity were challenging for the students. For example, when using the difference quotient to create a rate graph for the average rates of change of the derived function at integer x values and interval length 0.5 cm, many (78%) of the students plotted the interval width as 1 cm. This indicates that the key idea that average rates of change are always calculated over an interval was not yet fully understood by the students. The students had broad guidelines as to what to include in their reports. Hence, there were aspects of their models that were discussed by some students, but simply not addressed by others. For example, only some students (56%) provided arguments for why the first few data points had a measurement of 1 LIU; some students (22%) discussed sources of error that might have distorted the data during the measurement.

DISCUSSION AND CONCLUSIONS

The analysis shows how the students' diverse emerging models of how intensity varies with the distance from the light source converged as the students worked through the phases of the Light Lab. In line with the results of Whitney (2010), the students extensively used tables, but they also created contextually based graphical representations, from which the majority could successfully describe the data they collected within the context of the flashlight and light sensor. More than half of the students could meaningfully describe how the average rates of change of the light intensity varied. Using a meaningful context and actively engaging the students in the model application activity seemed to support some of the students in developing their models of changing phenomena.

The results of this study also suggest a number of issues, such as the meaning and importance of the interval width in the difference quotient and the validation of developed models, need to be addressed to support the continued development of the students' emerging models of average rate of change. Some of these issues could be addressed through a re-design of the activity. However, it should be emphasized that this model application activity was not the end of the model development sequence. Rather, it was followed by another model application activity that engaged students in examining the rate at which a fully charged capacitor loses its charge. The next step in our analysis of the development of students' emerging models will be an examination of that data. This study points to the potential value of examining the development of students' emerging models over sequences of modeling activities.

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