METAPHORICAL RANDOM WALKS: A ROYAL ROAD TO STOCHASTIC THINKING?

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We ask whether there is a "Royal Road" to stochastic thinking, which could be helpful and friendly to "regular", not mathematically oriented, students and teachers. We argue that random walks constitute such a royal road, since they allow a friendly visualisation of randomness, are amenable to experimental and enactive metaphorical study and provide "universal models" for a vast spectrum of stochastic problems. An explicit example of a didactical design is given (Brownie's walk) that has been tested with students and in service teachers of various backgrounds, since 2007. The findings suggest that this sort of approach significantly facilitate the access of not mathematically oriented learners to otherwise cryptic mathematical notions, allowing them to construct them while solving situated concrete problems.

INTRODUCTION

Our research question is: Is there a royal road to stochastic thinking?

Such a royal road would be especially helpful for "general" students in school, college and university who have other orientation than mathematics. For instance, students majoring in humanities and social sciences or prospective or in service elementary school teachers.

We claim that random walks constitute such a royal road. Motivation for this claim is both mathematical and didactical. We could also say that this claim comes from the no man's land between mathematics and didactics of mathematics: it does not seem to belong to the register either of a mathematician or of a mathematics didactician.

Mathematically, random walks constitute a universal model for a wide array of stochastic problems (see below).

Didactically, random walks have the virtue of being a concrete and iconic embodiment of randomness. They can be easily enacted, simulated and visualized by the learners. The study of simple random walks may be undertaken "bare handed", with practically no previous statistic or probabilistic tools or concepts. Students may tackle the paradigmatic question: Where is the walker, after a given number of steps? with sheer common sense. In particular they realize that there are several levels of answers to this sort of question, the 0^{th} level being: Nobody knows! For the subsequent levels see the example of Brownie's walk below. Random walks also

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provide a propitious soil for the emergence of metaphors that make possible the construction of the concept of probability: the probability of finding the walker at given position at a given time appears as the portion of the walker found at that position and time.

We rely on the metaphorical approach to didactics (Lakoff & Núñez, 2000; Sfard, 2009 ; Chiu, 2000; Soto-Andrade, 2012), which claims that our conceptual thinking is fundamentally metaphorical and that in fact metaphors are powerful cognitive tools that help us in grasping or building new mathematical concepts, as well as in solving problems in an efficient and friendly way. Metaphors play indeed various roles in our work: a "royal road" is itself a metaphor, a random walk may be a metaphor for a probabilistic problem (see below), a random walk may be metaphorized as an hydraulic flow (hydraulic metaphor) or as a deterministic progressive distribution of pedestrians along a road network (pedestrian metaphor).

In this paper, we describe a specific example of the use of random walks to introduce the students to stochastic thinking, that may be regarded as the germ of a didactical situation (Brousseau, 1998). This approach has been tested with students and teachers with various backgrounds (first year university students majoring in humanities and social sciences, prospective secondary school mathematics teachers, in service elementary school teachers, juvenile offenders engaged in a re-insertion program, among others). An a priori and a posteriori analyses are carried out, in the sense of didactical engineering. Finally, results obtained are discussed and some conclusions are drawn.

METAPHORS FOR STOCHASTIC THINKING.

Theoretical framework: Nature and Role of Metaphors in Mathematics Education.

It has been progressively recognized during the last decade (Araya, 2000; Bills, 2003; Chiu, 2000, 2001; English, 1997; Ferrara, 2003; Johnson & Lakoff, 2003; Lakoff & Núñez, 2000; Parzysz et al., 2007; Presmeg, 1997; Sfard, 1994, 1997, 2009, Soto-Andrade 2006, 2007, 2012, and many others) that metaphors are not just rhetorical devices, but powerful cognitive tools, that help us in building or grasping new concepts, as well as in solving problems in an efficient and friendly way: "metaphors we calculate by" (Bills, 2003). See Soto-Andrade (2012) for a recent survey. We use conceptual metaphors (Lakoff & Núñez, 200), that appear as mappings from a "source domain" into a "target domain", carrying the inferential structure of the first domain into the one of the second, and enable us to understand the latter, usually more abstract and opaque, in terms of the former, more down-to-earth and transparent. We notice than in the literature the terms representations, analogies or models are often loosely used as equivalent to metaphors.

We notice that Grundvorstellungen (better translated as "fundamental notions" than "basic ideas") for mathematical content, that are often operationally equivalent to conceptual metaphors, have been developed for a couple of centuries in the German school of didactics of mathematics (vom Hofe, 1995, 1998). In the case of probability they are usually given in a more prescriptive way than we do here for metaphors (Malle & Malle, 2003).

Our approach regarding the role of metaphors in the teaching-learning of mathematics emphasizes their "poietic" role, that brings concepts into existence ("reification" in the terms of Sfard (2009)). Also we posit (metaphorically) that metaphors are likely to emerge, as sparking voltaic arcs, in and among the learners, when enough didactical tension is built up in a didactical situation (Brousseau, 1998) for them.

BROWNIE'S RANDOM WALK

Instead of introducing first concept and tools that students will apply later to solving exercises and problems, as in traditional teaching, on the contrary, in the spirit of Brousseau (1998) we aim at students constructing or discovering those tools and concepts, when trying to solve situated problems, like a concrete random walk.

Perhaps the most natural visual example of randomness is Brownian motion. It was discovered rather late in the West, by British botanist Robert Brown in 1827, although it might be argued that is was "anticipated" by Lucretius in 60 AD (Powles, 1978) who observed dust particles under sunlight.

After presenting Brownian motion to the students in an interactive form, we try to settle for some "baby version" of it. Usually the students themselves have the idea of "simplifying" the phenomenon that we want to study, as much as possible, but without "throwing away the baby with the bath's water".

We consider here a specific "baby avatar" of Brownian motion: A 2D random walk, whose protagonist is a puppy, suitably called Brownie, sometimes suggested by the students themselves. Also an even simpler version, a 1D random walk appears, that we do not consider here. We discuss below, somewhat halfway between "bricolage" (Gravemeijr, 1998) and didactical design (Artigue, 2009), how this random walk has been tackled by the students with the help of suitable metaphors and without a previous knowledge of probability or statistics.

Where is Brownie?

Brownie is a little puppy that escapes randomly from home, when she smells the shampoo her master intends to give her. At each street corner, confused by the traffic's noise and smells, she chooses equally likely any of the 4 cardinal direction and runs nonstop a whole block until the next corner. Exhausted, after 4 blocks, say, she lies at some corner. Her master would like to know where to look for Brownie and also to estimate how far she will end up from home…

The mathematical situation (a description à la Brousseau).

"Arid" theoretical approach: We calculate stepwise the probabilities of finding Brownie at the different crossings in the city map. We obtain in this way the probability distribution of the sequence of random variables X_n = "Brownie's position after n steps" and D_n = "distance of Brownie to the origin after n steps". To obtain a general expression for this distribution, a clever idea is to do harmonic analysis and synthesis of these processes with respect to the (non-commutative!) symmetry group of the random walk.

Statistical approach: To see what is going on, we can make a statistical simulation of the random walk, eventually using a work sheet.

Metaphorical approach: With the help of an "hydraulic metaphor" (Soto-Andrade, 2006, 2007), we replace the puppy's random walk by a fission or sharing process on a grid. There we may see, for instance, the grid as a system of ducts and imagine a litre of water at the origin that flows symmetrically and fairly to the 4 next neighbours, each time. We have then to calculate stepwise with a deterministic process, which is equivalent to the original stochastic process, but has the virtue to avoid probabilistic language.

Didactical experimentation and methodology

Our research background consisted of:

a) primary school teachers enrolled in a 15 month professional development program, at the University of Chile, aiming at improving their mathematical competences, from 2007 to 2010 and in 2012.
b) 1st year University of Chile students majoring in social sciences and humanities, from 2007 to 2012 (1 semester mathematics course).
c) University of Chile undergraduates majoring in mathematics and physics (one semester probability and statistics course) in 2009-2011.
d) University of Chile prospective secondary school teachers (one semester probability and statistics course) in 2009 – 2011.
e) secondary school teachers enrolled in various professional development programs, in Chile, in 2006-2008 and 2011-2012
f) elementary school students (6 to 8th grade), whose teachers were engaged in the professional development program described above (in progress).
g) juvenile offenders enrolled in a social re-insertion program at the University of Chile, in 2012.

Learners were observed by the author and two assistants during work sessions (some of them videotaped), their written outputs were kept or scanned. They also answered questionnaires related to their use, appreciation and preferences regarding metaphors used in courses a) and b). Learners a) and e) did group work most of the time. Other learners participated in interactive lessons with a high degree of individual participation, horizontal interaction included.

We describe now the a priori and a posteriori analyses in the sense of didactical engineering (Artigue, 2009) related to this experimentation.

A priori analysis: Working in groups, the teachers or students should realize quickly that there are impossible corners (street crossings) for Brownie, even close to her home. There might be divergent opinions on whether returning home is possible after 1 or 3 blocks, for instance. Then, after having spotted the corners where it is possible to find Brownie (after a 4 block run), some (perhaps one half of the) students or teachers will believe that they are equally likely. Others would have the vague intuition that some corners are more likely because Brownie can get there by several different paths.

Quite late, some will have the idea of experimenting, by simulating a good number of puppies, to try to settle the question. The majority of them will now be convinced that some corners are more favorable to find Brownie; her home, for instance. They will not have much trouble in setting up a corner *ranking.* But they will have some trouble in quantifying their feeling of bigger or smaller likelihood. How to assign "weights" to the different corners? Some usually think of counting paths to arrive to quantify their vague feeling.

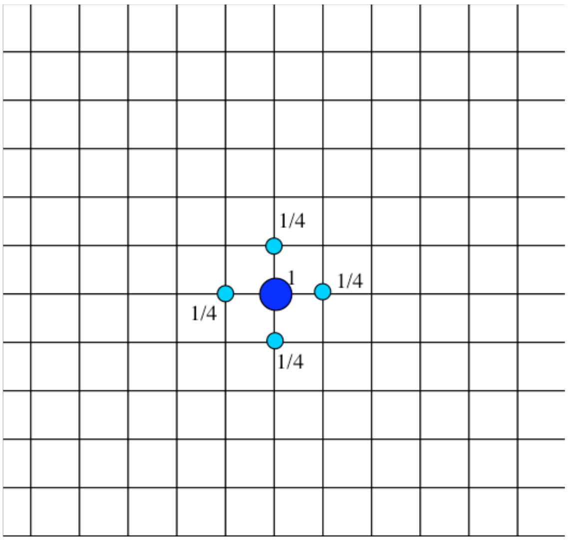
It is likely that the "solomonic metaphor" (cut the puppy into four pieces) or the "hydraulic metaphor" (the puppy flows equitably to the four immediate neighbours) will not emerge spontaneously. But this could happen under some minimal prompting like: What, more concretely, could you imagine instead of this (rather abstract) fair random choice between the 4 cardinal directions? (using some discrete gesture language). The students or teachers should not have much trouble quantifying the likelihood of presence of the puppy at the different corners, with the help of the hydraulic metaphor. They will realize quickly the conservation law of the puppy: putting together her pieces at each step you should be able to reconstruct the whole puppy.

The idea might also emerge in the classroom to unleash a pack of puppies from home, that would spread out evenly, splitting into four equal groups at each corner. Better begin with 16 puppies which will run 2 blocks each…

A posteriori analysis: We did not foresee that the primary school teachers would have a strong tendency to see without hesitation the 9 possible corners for Brownie after a 2 block run as equally likely! Dissenting opinions emerged very slowly among them. On the contrary learners from groups b), c), d) quickly sensed that corners closer to her home at more likely.

We have witnessed the emergence in the classroom of metaphors like the solomonic, hydraulic or pedestrian one (Soto-Andrade, 2006, 2007) right after the students or teachers were prompted to try to see the situation "otherwise", while working in groups, with discrete support from the facilitator. Typically in a class of 30, one or two immediately "see" the puppy split into four. These metaphors emerged more easily from learners in groups a) and b). Initially however many teachers seemed to have feeling that they were violating some (unspoken) didactical contract when allowing themselves to metaphorize.

In cases where this sort of visualization seemed out of reach, the hydraulic metaphor has enabled us to act out the situation: 13 in service elementary school teachers distribute themselves suitably on the nodes of a virtual grid on the floor, one of them has a container with 1 litre water that she shares equitably with her immediate 4 neighbours, and so on…Each one of them could estimate the amount of water that she will have under a given number of steps. Eventually they simulate this (more easily although less dramatic..) with a square of board that they partition into four pieces and so on. See figures below.



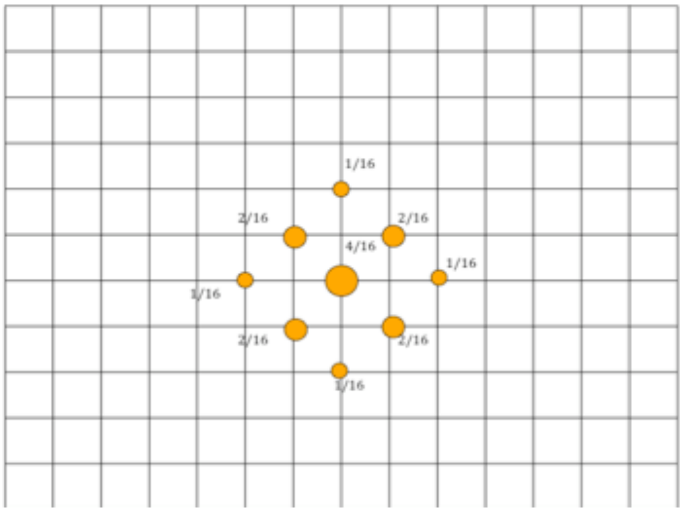


Fig. 1 Brownie's splitting (0 and 1 block). **Fig. 2. Brownie's splitting (2 blocks).**

Notice that in this acting out, each participant does something very simple, but the outcome is the analogical solution of a non trivial problem. This could seen as a just one simple example of application of "swarm intelligence" to the didactics of mathematics.

With this course of action, in our view, the students or teachers, may succeed in constructing the concept of probability at the same time that they solve the problem in context, with the help of suitable conceptual metaphors (Lakoff and Núñez, 2000; Soto-Andrade, 2006,2007, 2012; Sfard, 2009).

We remark that the role of the teacher in this sort of "mise-en-scène" turns out to be quite delicate. Definitely better than the metaphor "a teacher is a technician" emerges here the metaphor "a teacher is a tightrope walker". In particular, experience shows that didactical micro-gestures of the teacher - tightrope walker may feed a butterfly effect in the classroom.

Notice that in our metaphoric approach the students are not given a "Grundvorstellung" (fundamental notion) for probability before they address the problem. On the contrary, they are prompted to tackle the problem "bare handed" first and eventually look for a friendly metaphor for the *concrete random walk* they want to study (e.g. "Brownie splits"). When trying to give pertinent answers to the questions asked, "poietic metaphors" may emerge that enable them to construct the abstract probability concept, like: "probabilities of finding Brownie at a given corner are pieces of Brownie". This can be seen as a germ of an adidactical situation in the sense of Brouseau (1998).

Brownie's walk seen from Google Earth (zooming out).

First, Brownie's walk can approximate efficiently Brownian motion: just make the grid denser and denser…If you zoom out, with the help of Google Earth, tracking Brownie with GPS, you will see the trajectories of pollen grains.

Second, our title is also metaphoric: if we zoom out cognitively, we realize that Brownie's random walk is a paradigmatic example of a random walk (a metonymy, in fact) and that random walks play very often the role of universal models in probabilistic as well as statistical problems. For example:

1. The famous Italian unfinished tournament problem: Two players of equal strength compete in a tournament that consists of a series of games. The winner, who will get the one million euros prize, is the one who completes 10 wins first. Now, the tournament must be interrupted by "force majeure" when one player had won 8 times and the other 7 times. How can the prize be divided fairly between the two players?
This problem can solved very easily by modelling or metaphorizing it by a random walk (a 2D one!) and solving the random walk metaphorically. This is friendlier than Pascal classical solution (apparently Pascal was not very fond of random walks…) If we recall that "mathematics is the art of seeing the invisible" (Soto-Andrade, 2008) we would agree that what the students are doing, when they see the development of the tournament as a random walk on a 2D grid with absorbing barriers, is real mathematics.

2. Evaluating screening tests (e.g. false positives problems) is a tough task for experienced physicians (Zhu & Gigerenzer, 2006; Gigerenzer & Hoffrage, 2009; Gigerenzer, 2011). The typical question that physicians answer wrongly is: "I have got a positive HIV test, how likely is that I am really a carrier?"
Natural frequencies have been suggested (loc. cit.) as a means to get a correct answer without much toil, in a way a 10 year old could do. But if you look at this problem as a question about a (2 step) random walk, that you can solve with the help of a pedestrian metaphor, your recover exactly natural frequencies.

It is interesting to compare the relative popularity, among students in various levels and countries, of the hydraulic and pedestrian metaphors. According to Gigerenzer (loc. cit.), pedestrian metaphors (i.e. natural frequencies) should be much more popular, because you just manipulate whole integers and you compute a fraction only at the very end, or even you get your result in the form "m out of n", as in pre-fraction days, that is all the same efficient for practical purposes. In the hydraulic metaphor however you need to manipulate fractions all the way. We have found experimentally however, consistently year after year, that 1st year university students majoring in Social Sciences and Humanities (group a) above) tend to prefer the hydraulic metaphor (8 out of 10 approx.), even if computing with fractions is not all that smooth for them. Apparently one reason for that is the *conceptual* impact of seeing one litre of grapefruit juice (or probabilistic fluid, if you like) draining downwards through a graph of ducts, something that helps them tackle Zeno's paradox, for instance.

3. The classical didactical situation of Brousseau (1998) where the 7th graders try to find out the composition of a bottle containing 5 marbles of 2 different colours by looking inside through a tiny opening (that lets them see just one marble at a time), can be modelled by a random walk in the plane that "ends up" engaging into *one* region out of four possible wedge-like regions (corresponding to the 1-4, 2-3, 3-2 and 4-1 compositions)

DISCUSSION AND CONCLUSIONS.

Our research question was: "Is there a royal road to stochastic thinking". We claimed that there is at least one, to wit random walks. So we can access stochastic thinking "random walking".

We described an explicit didactical engineering for a concrete example of a random walk (Brownie's walk) whose "bare handed" study facilitates the construction of the concept of probability, thanks to the likely emergence of various helpful (mainly enactive) metaphors.

We tested this didactical engineering with learners of several backgrounds, students as well as in service teachers, across the country (Santiago, Copiapó, Puerto-Montt, Talca, Parral). According to their performance (assessed by tests and group work) and answers to questionnaires and interviews, this approach (random walks metaphorically tackled) did help them to understand otherwise cryptic mathematical notions and to solve problems involving randomness in a friendly and efficient way (e.g. false positives problems). We notice some initial resistance to metaphorizing, especially among university students majoring in mathematics and secondary school teachers. University students majoring in humanities and social sciences and primary school teachers were more prone to metaphorizing, especially after being granted permission to do so ("you are not supposed to metaphorize in mathematics", seems to be the implicit current didactical contract). Among university students remarkable metaphorical capacities were detected in students coming from alternative schools, like Montessori or Waldorf. Otherwise it seems clear that traditional teaching in Chile, especially from grade 7th onwards tend to thwart metaphorizing. This suggests promoting this approach already at the beginning of the elementary school and exploring or constructing other royal roads. Further research on this didactical phenomenon quantitative as well as qualitative seems advisable.

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