THE DIDACTICAL KNOWLEDGE OF ONE SECONDARY MATHEMATICS TEACHER ON STATISTICAL VARIATION[1]

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This study aims to contribute to the comprehension of the didactical knowledge of the teacher who teaches statistics at the secondary level, looking at the classroom practices of one teacher regarding statistics variation. Results stress the intertwined character of the various domains of the teacher's didactical knowledge, giving evidence of a strong relationship between the teacher's knowledge of statistics variation and the level of depth by which she approaches variation in the classroom.

INTRODUCTION

Today's growing importance of statistics has called attention to its relevance in education, giving it a larger prominence in national and international school curricula. Recent research points out the emergency of a set of fundamental big ideas for the teaching of statistics, such as data, variation, distribution and representation, among other concepts, which every student should know at the end of secondary school (Burrill & Biehler, 2011). In fact, there is an increasing recognition that variability is a fundamental idea for the teaching of statistics (Shaughnessy, 2006; Garfield & Ben-Zvi, 2008). The GAISE report (Franklin, Kader, Mewborn, Moreno, Peck, Perry & Scheaffer, 2007) provides a framework model for statistical education over three developmental levels in which the focus on variability gets more sophisticated as students move from one level to a higher one. Wild and Pfannkuch (1999) emphasized the importance of variation in statistics considering that it is "the reason why people have had to develop sophisticated methods to filter out the messages in data from the surrounding noise" (p. 236). However, several studies also suggest that variability is a difficult concept for students to reason about (Garfield & Ben-Zvi, 2008). According to these authors, such difficulty does not rely on computing formal measures of variability (e.g. range, inter-quartile range and standard deviation), but rather on understanding different representations of the concepts involved in those computations as well as their relationships with other statistical notions.

Teachers should help students to build on their intuitive notions of centre and variability and integrate these concepts when addressing data and doing data analysis as suggested by Shaughnessy (2006). Teaching statistics does require a profound content knowledge of this area; however, such knowledge is not enough to ensure effective teaching (Ponte & Chapman, 2006). Sánchez, Silva and Coutinho (2011) suggested the study of teachers' professional knowledge while teaching variation as an urgent matter. Thus, it is important to analyse teachers' didactical knowledge in statistics, in particular of this topic, considering their perspectives on the teaching and learning of statistics and the ways it is approached in the secondary classroom. This study, which is part of a larger investigation, intends to make a contribution to this call, aiming at analyzing one secondary mathematics teacher's didactical knowledge focused on the concept of variation.

BACKGROUND

Didactical knowledge in Statistics

According to Ponte (1999), a teacher's didactical knowledge comprises four domains that orient directly the school practice. Each domain is here described in the realm of statistics, looking for enhancing its specificity.

The first domain is the *knowledge of the content* to be taught. It includes understanding concepts and procedures, statistical reasoning, argumentation and validations forms. This knowledge also includes the ability for epistemological reflection about the meaning of concepts and procedures and about the nature of statistical knowledge (Batanero & Godino, 2005). In order to teach statistics, the teacher must have experience and be familiar with specific elements of this subject, namely, the fundamental aspects of statistical thinking, such as recognizing the need of data, attending to variation, among others (Wild & Pfannkuck, 1999).

A second domain is the *knowledge of students and their learning processes*, which incorporates the knowledge of their difficulties, mistakes, obstacles and typical strategies in problem solving. In this domain, it is also important that the teacher have a perception of the level of understanding that students have or will attain regarding different concepts (Ponte, 1999; Batanero & Godino, 2005).

The *knowledge of curriculum* includes the knowledge of curricular goals and objectives, and the horizontal and vertical articulation/alignment of topics (Ponte, 1999). The current curricular framing enhances the active role of the students in building their mathematical and statistical knowledge (DES, 2001).

The fourth and last domain of teachers' didactical knowledge is the *knowledge of the instructional process*, which includes planning, teaching of lessons and assessment of practices (Ponte, 1999). It is through this knowledge that teachers organize their own practice and give answers to the various situations of interaction with students. This domain also encloses the capacity of adapting the subject matter to be taught along the various teaching levels, of adjusting the instructional process to the interests, attitudes, and conceptions of students, of analyzing several methodological resources and of communicating with students in the classroom.

The Teaching and Learning of Statistical Variation

The fact that many mathematics teachers have a limited training in statistics and have not experienced data analysis makes it difficult to develop their statistical thinking, which has effects on their own practices (Ben-Zvi & Garfield, 2004). Indeed, more often than not, teachers focus on computations and procedures rather than on data analysis and interpretation of results (Rossman, Chance & Medina, 2006). Sánchez and colleagues (2011) still add that teachers do not give enough attention to explanations and adequate use of terminology. Although acknowledging the usefulness of technical procedures, Scheaffer (2000) stresses the need for a genuine "understanding of analysis and communication of [statistical] results" (p. 158).

Wild and Pfannkuck (1999) assume that the *consideration of variation* is one of the principal aspects of statistical thinking and describe it as the capacity of measuring and modelling variation and making decisions from data. It also involves looking for and describing patterns in the variation and trying to understand these with respect to the context. The term *variation* describes or measures the change in the variable, while the term variability refers to the phenomena of change (variable) (Reading & Shaughnessy, 2004). In this paper these terms are used indistinguishably.

Several studies state that students face serious difficulties in perceiving some measures of variability. In order to help students developing their reasoning about variability more efficiently, Garfield and Ben-Zvi (2008) suggest that their initial statistical activities should move from understanding informal ideas (e.g. differences in data values) to understanding and interpreting formal ideas of variability (e.g. standard deviation as measure of average distance from the mean; factors that can cause standard deviation to be larger or smaller). The concepts of distribution, mean and deviation from the mean are essential for reaching a meaningful notion of standard deviation (Sánchez et al., 2011) which, in turn, is important for understanding variation. The GAISE report (Franklin et al., 2007) indicates some transitional interpretations for the mean (numerical summary of centre for a set of numerical data) to be developed in school, namely, the mean as the fair *share value* for data in a lower level of comprehension, or as *a balance point* of the corresponding distribution in an upper level. It is crucial that teachers help students to build on their intuitive statistical concepts toward a more sophisticated understanding.

METHOD AND CONTEXT

This study is part of a larger ongoing qualitative research that focuses on Portuguese secondary mathematics teachers, more concretely on these teachers' didactical knowledge of statistics. This communication refers to Estela, one of the three case studies of the wider research, currently being the more developed one. The three teachers have in common the fact of: being experienced teachers; each of them teaching a 10^{th} grade class in the 2010/2011 school year; their teaching schedules allowed for observing all classes, and they demonstrated interest in participating in this investigation. Estela is a certified mathematics teacher and has a master degree in mathematics education. She has been teaching mathematics for 23 years.

The data that informed the present study were collected using several instruments, namely: (1) participant observation, with audio and video recordings of eight lessons on statistics taught by Estela to a 10^{th} grade mathematics class with 25 students; (2) three semi-structured interviews to Estela with audio recording; and (3) documental collection of the resources used by the teacher in the observed lessons, especially working sheets. The data analysis was accomplished in a descriptive and interpretative way, pondering and articulating the various collected elements. This analysis was guided by four assumed pre-categories that correspond to the four domains of the teacher's didactical knowledge of Ponte (1999).

The environment in Estela' class can be characterized as informal, with a good pace of work and a positive interaction between teacher and students and among students. The concepts or topics to be taught in class are often introduced through a task. Most of these tasks are selected by Estela from different textbooks and only a few are adjusted or created by her. Students usually work in pairs and the task's correction is mostly performed by the teacher in interaction with some students, once she realizes that most students have completed its resolution.

In her teaching, Estela followed the sequence of contents suggested in the secondary mathematics curriculum (DES, 2001), which is divided in three parts with the following order: (1) generalities about statistics (historical evolution, aim and utility; population, sample, sampling - some intuitive ideas; the statistical procedure: descriptive and inferential statistics; (2) organization of quantitative and qualitative data (process of summarizing data sets numerically and graphically); and (3) introduction to the study of two-dimensional distributions (graphical and intuitive approach).

RESULTS

Perspectives of the teacher about statistics and its teaching

According to Estela, the 10th grade mathematics curriculum regarding statistics focuses mainly on descriptive statistics, that is, on the summarization of quantitative and qualitative data through the use of multiple representations, giving little attention to inferential statistics and to interpreting results. Estela considers that the fact that statistics is not an object of assessment on national examinations and has no continuity throughout the remaining secondary grade levels contributes to a smaller importance given by teachers and students in comparison to other curricular topics. She believes that statistics is a topic teachers do not fully master. Having in mind the computation of standard deviation on a graphical calculator, she mentions that

> Because sometimes it shows up in the graphical calculator… when I'm computing standard deviation S and Sigma appears on screen. You [may] remember that two or three years ago in a vocational mathematics exam there was a big issue because students were choosing the σ value [population standard deviation]… but since the task stated it was a sample then they [students] had to consider S [sample standard deviation]. And whose teachers were aware of it? No one. (…) Do you know where I heard that? In my master degree, never in college [during undergraduate studies]! (…) And nowadays they [teachers] reach it [this distinction]. Textbooks didn't.

Estela only recently learned that after making the computation on the graphical calculator for obtaining the standard deviation value, she has to choose one of the two possible results. That is, this choice depends on whether the data set corresponds to a sample or a population. In her opinion, this issue was not highlighted in school textbooks and moreover is not part of teachers' usual content knowledge.

Estela thinks that the statistics in the secondary mathematics curriculum emphasizes, above all, its descriptive side, where computation and procedure repetition prevail, with little space for carrying out interpretations and reaching conclusions. Thus, in her practice, the tasks that she often asks students to work on contain several subsections, each of them very directed and with little focus on interpretation:

> The questions [in statistics] have to be asking more things (…) and are very oriented to answer something, in Mathematics it is just that something, it is not for discussing… Since when? ... Mathematics is exact and it's over!... and statistics is the same… it is for reaching that something and nothing more!

Referring to the curriculum, Estela says that "There is almost no inductive statistics in here!", considering that it calls very little to make inferences and does not value the generalization over a population based on a sample. Furthermore, she says that:

> That's why students tell me "Teacher, I can't wait to get started with statistics". Because they already know most of it [the contents]. What is really new about it? Maybe the spread measures and the two-dimensional variables (…) Also [the boxplot], but now in the new [basic education] curriculum it's there. It makes me wonder. (…) When the new generation in the 8^{th} grade gets to 9^{th}, there will be a new 10^{th} grade curriculum because the diagram of extremes and quartiles won't be taught if they've had it in the 8^{th}. Will this be a factor for a change? Will there be room for working more on inferential statistics?.

Through this excerpt, Estela enhances the need of a new statistical curriculum at secondary level that takes into account its inferential side, since the majority of the issues approached for the first time at this level were recently enclosed in the mathematics curriculum for basic education.

Since the graphical calculator is a mandatory resource at secondary level, Estela thinks that one must take as much advantage as possible from this device, particularly useful in students' learning, namely when working on a task with a lot of data, releasing students from the burden of extensive computations. In her perspective, this resource can also help to promote teachers' statistical knowledge for those who lack preparation in this field, allowing them to develop some skills in statistics, as it generates additional questions about results and representations:

> Since now they are not required to know how to compute those formulas with a lot of data, computing the mean is unthinkable, computing the median is unthinkable, the quartiles, the standard-deviation. (…) I'm still from a time when the graphical calculator wasn't mandatory in school and of making huge handmade tables on the blackboard, it doesn't make sense [anymore]… they [the students] found them very difficult and I thought it was the easiest part… I'd start with the mean, then would do *x* minus the mean… it was very easy and the calculator [nowadays] does it easily. And with the

> calculator we can have more data and even us teachers who didn't have this teacher preparation [in statistics]… for us to learn and develop more ideas and skills about statistics… it may generate more discussion and one can do more than trivial things.

The Study of Variation in the Classroom

The concept of (arithmetic) mean was mentioned for the first time in class by a student who referred to it describing its computation formula. And it was only in this way that the concept of mean, often targeted in the observed lessons when solving several tasks, was approached by students and by the teacher. It is a value that was most of the time computed by a graphical calculator (TI-84 plus) after students inserted the list of numerical data and followed the instructions: *Menu Stat* - choose *Menu CALC* - choose *select option 1: 1-Var Stats* with the corresponding data list.

However, in one of the interviews, while reflecting on some informal ideas attained by students concerning the mean, Estela mentions its visual representation, considering that the mean may be more difficult to be visualized in some graphical representations (histograms, bar diagrams) than the median:

> Since the mean is very influenced by outliers but I believe that they [the students] were able to more or less figure out if the mean was on the left or on the right of the median.(…) The intuitive idea is trying to approximate the mean and the median, but sometimes it is not [possible], just when the [data] distribution is symmetrical. This is the idea that I have about what they [the students] got [about the mean].

In this excerpt, when mentioning the visualization of the mean graphically, Estela seems to convey that the fact that there may exist outliers in the distribution may affect the symmetry, and that the visualization of the mean by the students, in this case, may not be so easy. However, she considers that students would not have difficulties in finding the mean in case the distribution of data were symmetrical.

Regarding the concept of standard deviation, students got used to seeing its symbol on the graphical calculator, when they obtained summary statistics for a numerical data set (discrete or continuous) before learning it in class. For Estela, knowing how to calculate the standard deviation by hand, besides being a lengthy procedure, is not a curricular goal. When talking about this concept in class, she shows its formula and mentions that it is "the positive square root of variance". Estela defines it as a spread measure that "measures how far the data is from the mean" and exemplifies on the whiteboard how students should compute this measure with a set of numerical data. Yet, she also says that students must determine this value using the graphical calculator, following the instructions: *Menu STAT* - choose *Menu CALC:* choose *option 1: 1-Var Stats* and add *List*, whose corresponding value is found in *S* or in σ.

With the intention of showing a different way of interpreting standard deviation, Estela talks about the confidence interval "mean *more* or *minus* standard deviation that contains approximately about 68% of data…". She checks together with students the veracity of this statement, reaching the conclusion that in fact 71% of the data were included in the referred interval. Additionally, she alerts very quickly that this distribution of data should be of a “particular shape” in order for this property to hold, saying that it was something to be learned with much more detail in college. In Estela’s opinion, the intuitive ideas that her students got about standard deviation could be summarized as: “given two sets of data values with the same mean… the one that has larger standard deviation value is the one where data is more spread out [with respect to mean]”. This excerpt seems to indicate that this teacher considers that students take standard deviation as a spread measure. Her intuitive idea is correct, as long as none of the distributions have outliers.

In one class, Estela proposed a task that she designated by “Properties of the mean and standard deviation” (see figure 1 below):

In a company the wages in the year of 2005 were the following:

Wages (euros)	600	700	800	1000	1500	2000	4000
Absolute Frequencies	20	50	15	9	3	2	1

a) Compute the mean, mode, median, quartiles and the standard deviation of this distribution;
b) Supposing there was a 5% wage raise in 2006, what would be the wages in 2006?
c) Compute the mean, mode, median, quartiles and the standard deviation of the 2006 wage distribution with the 5% raise. What do you conclude, comparing with the results from a)?
d) Supposing there was a 50 euro raise in 2006, what would be the wages in 2006?
e) Compute the mean, mode, median, quartiles and the standard deviation of the 2006 wage distribution with the 50 euro raise. What do you conclude, comparing with the results from a)?

Figure 1: Task about the properties of the mean and standard deviation

Estela thinks that this task, proposed in a working sheet, was very successful in promoting the study of those two concepts starting from the very same problematic set up; this task design was quite different from the tasks’ she found in textbooks.

The interaction with students went on in the following way. After having read the task, Estela asked students to make all the requested computations at once on the graphical calculator. Next, she reminded students that they should insert two different lists of numerical data (one corresponding to the wages and the other to their respective absolute frequencies). Estela showed how to proceed with the calculator. Several students revealed difficulties in understanding why they had to multiply by 1.05, instead of multiplying by 0.05 for wages to increase by 5%. Estela spent some time trying to clarify this issue. After having made all the initially requested computations writing them on the whiteboard – simultaneously with students – and having pointed out the influence of making a specific percentage or constant growth in the data values and their effect on each statistical measure, Estela concluded the task in the following way:

> So, the story’s moral: (…) if I multiply the values of the statistics variable by a number, all the statistical measures become increased by that number, increased…, I mean…

multiplied. But if I add up a number to the statistics variable, all the values of the statistical measures become increased by that number, except for the standard deviation that remains unchanged. Is it understood?

The development of this task was quite centered on the computation of the statistical measures using the calculator and on the comparison of the different attained results as the initial data was being changed. This task was not restricted to the proprieties of the mean and standard deviation since the mode and quartiles were also compared.

Neither the teacher nor any student questioned any obtained result, in particular, why the behavior of the standard deviation was different from the other statistics measures when the initial data was increased by 50. This situation may suggest an overvaluation of the results from the graphical calculator, which may have hindered from wondering why the standard deviation remained unchanged. The understanding of the properties of the mean and standard deviation seems to be considered important by the teacher for a deeper knowledge of these statistical measures.

DISCUSSION

We now analyze the dimensions of Estela's didactical knowledge of statistics, with particular incidence in variation, emerging from her perspectives and classroom practices. Estela shows evidence of owning a *knowledge of the curriculum* restricted to some aspects with particular incidence on descriptive statistics. Such knowledge was noticed in her practice, for example, when working on the concept of mean, focusing mainly on its computational side (Rossman et al., 2006) and, in particular, not mentioning explicitly any of the mean interpretations referred in the GAISE report (Franklin et al., 2007).

The *knowledge of the students and their learning processes* as a result of her statistics teaching experience gives her the conviction that students have more difficulties in seeing the mean graphically than the median. However, Estela thinks that in the presence of a symmetrical graph or an approximately one, students know that the values of the mean and the median are close. Also, concerning standard deviation, she considers that students are able to identify it as a spread measure, taking into account the residual values with respect to the mean value. Ponte (1999) and Batanero and Godino (2005) pointed out that teachers need to be aware of their students' reached level of comprehension of the concepts, as well as difficulties, in order to improve their teaching and assessment practices.

Estela's work in the classroom on an application of the standard deviation concept and the search for a confidence interval seem to indicate that, from her point of view, the topic of statistics should be expanded in terms of content, encompassing more inferential statistics. The same happens when she refers the need to develop the secondary curriculum given the recent curricular changes in basic education, which started to incorporate much statistics content currently addressed at the secondary level. However, in the task that she called "Properties of the mean and standard deviation", which had potential to help students to develop their reasoning about centre and variability, Estela ended up giving work orientation much centred on computations using the calculator, and in this interaction, neither the students nor the teacher questioned the obtained results. This situation suggests a certain overvaluation of the calculator results, since the need of reinforcing the interpretation and explanation of results was not observed. This is a commonly found phenomenon in mathematics teachers (Rossman et al., 2006; Sánchez et al., 2011) and resonates with Estela's interpretation of the written curriculum. Furthermore, this interaction did not bring along any conjecture to be pursued. In fact, there are other properties, regarding variation, that could have been object of reflection in this or other tasks, namely that the median is a value located between the extreme values of the distribution, or that the residual sum with respect to mean is zero. Nevertheless, these situations did not occur in Estela's classroom, raising important questions concerning her *knowledge of the instructional process* in teaching variation.

In her instructional process, Estela seems to give some relevance to the deepening of the mean and standard deviation concepts, since she proposed a task in this direction; yet, the understanding of formal ideas of variation as suggested in the literature (e.g. Garfield & Ben-Zvi, 2008; Sánchez et al., 2011) does not seem to be totally achieved. This situation may be related to Estela's formal comprehension of the standard deviation concept as noticed while presenting its definition in class or when reflecting on the informal ideas that students may hold about standard deviation. For instance, she did not contemplate all the hypotheses that would make it always valid. Her *knowledge of the content in statistics* seems to affect the level of depth in which these concepts were developed in the lessons.

This analysis of Estela's practice gives evidence that the various domains of the teacher's didactical knowledge are interrelated, and consequently that they must be taken into account in the study of the teachers' practices and also in teachers' preparation and professional development programs.

NOTES

1. Work funded by FCT – Fundação para a Ciência e a Tecnologia, PORTUGAL, under the scope of the project *Developing statistical literacy: Student learning and teacher education*, grant # PTDC/CPE-CED/117933/2010

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