WHAT DOES IT MEAN TO DO STOCHASTIC? IDEAS, SYMBOLS AND PROCEDURES FOR RANDOMNESS

Chiara Andrà[1], Judith Stanja[2]

[1]University of Torino, Italy; [2]University of Duisburg-Essen, Germany

We use Rotman's terms of idea, symbol and procedure to propose a frame for characterizing stochastic thinking. Stochastic uses specific artifacts but also artifacts from other domains of mathematics. The aim of this work is to apply Duval's model of coordination of semiotic means for studying the procedure component of the Rotman-frame. As an example, we give a qualitative analysis of a questionnaire given to a pre-service teacher. The questionnaire addressed the interpretation of artifacts that may be part of elementary stochastic. The result of the analysis suggests that stochastic requires a specific usage and interpretation of artifacts. The final discussion concerns practical issues, the potential of a semiotics perspective and methodological issues.

Keywords: elementary stochastic, signs, artifacts, semiotics, stochastic thinking

INTRODUCTION AND BACKGROUND

In his work, Rotman (2003) investigates what it means *to do mathematics*, and states that "behind the various construals of mathematics as an activity […] lie three distinct, fundamental theoretical discourses that enter into the subject, namely: idea, symbol, and procedure." (Rotman, 2003, p. 1676). *Idea* stands for the domain of human thought, as delineated by the individual's narratives in natural language. Within the frame of stochastic, *Idea* can be understood as the domain of the intuitive approaches to uncertain situations (Andrà, 2011). According to Andrà and Santi (2011), a person intuits mathematical concepts when the access to the distinctive features of the mathematical object is self-evident, coercive, and global. Referring to Radford's (2008) perspective, Andrà and Santi claim that self-evidence and immediacy can be traced back to "the spatial-temporal, sensorimotor and perceptive activity that semiotic means of objectification accomplish, support, foster" (p. 115). In a previous study, Andrà (2011) identified three different levels for approaching probability: the level of perceptive and sensorial experience, the level of arithmetic thinking, and the abstract, ideational, theoretical level. Artifacts, which are part of activities and sense making processes exist in time and space. The development of personal and collective practices for productively using the artifact as an instrument in knowledge-building activity represent a shift to another level – the S*ymbol*, which is the domain of signs, as well as communicational and semiotic practices from notational devices to entire linguistic systems. Within the frame of probability, *Symbol* is the set of signs and symbols that are commonly used in probability (Venn diagrams, formulas, tables, histograms, and so on). The artifacts change their meaning at the level of S*ymbol* – they become signs. A sign can be meant as an artifact that is connected with a meaning (or used as representation for a mathematical idea). In order to see an artifact as a mathematical sign, it needs to be related to a mathematical idea (see also Stanja, 2012). Finally, *Procedure* is the domain of actions, processes, and operations on and with artifacts. According to Rotman, for thinking mathematically all three – ideas, symbols and procedures have to be coordinated.

The aim of this contribution is to attempt a first step in order to show that the coordination of semiotic means (Duval, 2008), which need to be done when doing stochastic differ from coordination in other domains of mathematics, and may therefore be useful aspects for characterizing stochastic thinking. Stochastic education literature provides several frames for probabilistic thinking (see for example Jones et al., 1997) or statistical thinking (see for example Wild & Pfannkuch, 1999). For this contribution, we understand stochastic as part of mathematics including both probability and statistics. For stochastic education research the introduction of stochastic at the Primary school level raised new (and old) questions and problems concerning the nature of stochastic and the learning and teaching of stochastic. The limitations concerning semiotic means at the Primary school level motivate the consideration of semiotic perspectives. The specificity of this work can be traced back to the use of Rotman's frame to study the semiotic coordination of artifacts and signs, in a task regarding the perception of randomness.

Stochastic as cultural product: a semiotic perspective

We understand stochastic as part of mathematics and therefore historically and culturally determined. There is consensus that stochastic needed some time to develop and changed in its development (for a historical examination of the development of probability we refer the reader for example to: Barbin & Lamarche, 2004). If we look at stochastic from this perspective, then artifacts play an important role in the mediation and communication of stochastic. As in other domains of mathematics the objects may not be perceived directly, they can only be accessed via semiotic means (Duval, 2008). To represent abstract stochastic objects in an adequate way a variety of semiotic means needs to be coordinated. (Compare also Duval, 2008 and Rotman, 2003). These coordinations may be described by the terms *treatment* and *conversion* (Duval, 2008). *Treatment* refers to transforming a representation into another one within the same semiotic system. Calculations are an example for a treatment in the system of fractions. A *conversion* means a coordination of representations of different semiotic systems like a verbal description of a list (lists → language). *Conversions* can be further distinguished into *congruent* and *non-congruent* conversions. *Non-congruent conversion* means that no 1-1-mapping between all meaningful components of the representations in question is possible. The choice of meaningful components of the target representation is ambiguous. A meaningful component in one representation may not have a matching counterpart in the components of the other representation. Furthermore, for all relevant components that can be mapped, the relation between the components (structure) may change. *Congruent conversion* means that a 1-1-mapping of the meaningful components is possible, the choice of these components in the representation is unambiguous and the structure between the components does not change. As claimed in Stanja (2012), the way a learner coordinates given artifacts may give substantial insights in the development of the stochastic thinking of the learner. Ben-Zvi *et al.* (2011) confirm the importance of tools as supporting elements for informal inferential reasoning processes. Arteaga and Batanero (2011) relate the semiotic complexity of graphs to the ability of reading between and beyond data, and conclude that there is a strong relationship between the ability to *read* a graph and the access to the more sophisticated and abstract activities of *interpretation* and *extrapolation*.

MARTA'S EXAMPLE

Background Information

As an illustration of theory, we now present the written example of Marta, a pre-service primary school teacher. She received the questionnaire (figure 1) at the very beginning of a course in probability in the Academic year 2011/12 at the University of Torino. Participants of the course were attending the third year of the undergraduate course in Primary School Education and had no experience in stochastic so far. As well, they had no teaching of probability/statistics during the school period. We were interested in how the pre-service teachers would spontaneously work on this kind of problems and what possible obstacles may be found concerning the usage and interpretation of the given means.

The questionnaire was designed by Andrà and is based on interviews that had been carried out by Stanja in the school year 2010/11 with grade-3 students in Germany. In the first task, the students were given an artifact, a spinner (Figure 1), they were asked to give a prognosis about the possible outcomes when turning the spinner 20 times and to give a justification. In the second task, the relational focus of spinner to outcomes (represented by the lists) was reversed. There were four different sequences of 20 trials shown and the students were asked to comment on the results. In the third task the students were asked to draw a sequence they imagine to obtain by turning the spinner 20 times and give an explanation of their action. All tasks had in common that they demanded the usage/coordination and interpretation of given socially constructed means the students were not familiar with but that could be part of an elementary stochastic course. The questions were open concerning the way of representing the outcomes and allowed to answer in more or less details. A similar task has been considered by Batanero, Godino & Roa (2004), in a study aimed at describing the knowledge needed for teaching probability at school. Specifically, the task pivoted around the concept of randomness, and the researchers observed that different students pay attention to different properties of the sequences (analogous to question 2 here). All in all, they observed the emergence of common heuristics such as the negative recency, and representativeness (Kahnemann *et al.*, 1982). In other cases, they observed that the students reason according to the outcome approach (Konold, 1989).

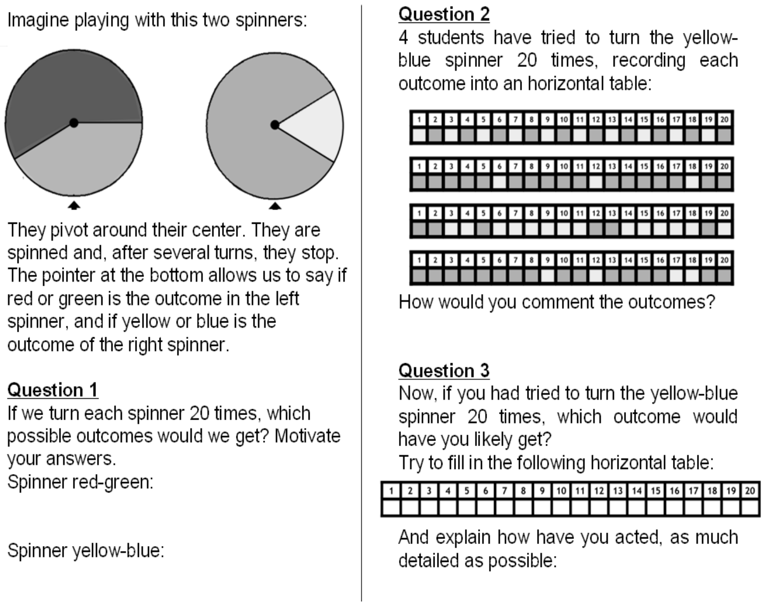


Figure 1: the English translation of the test given to the students.

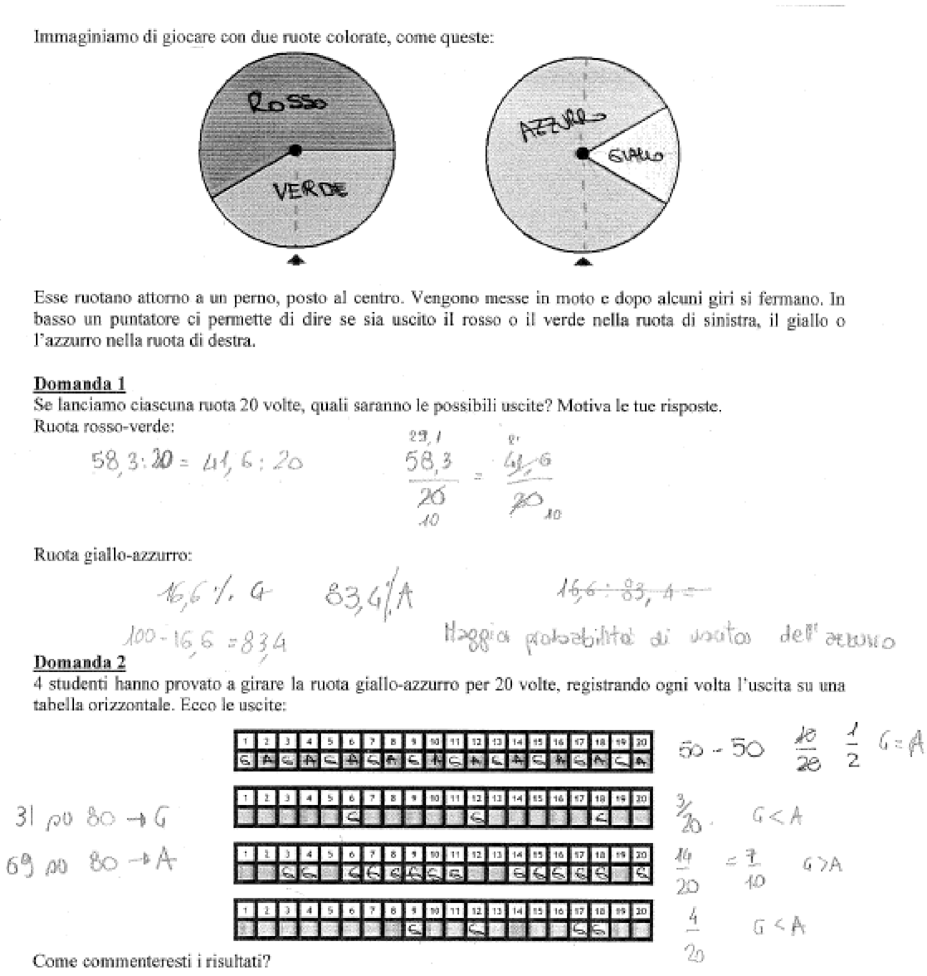
We give an interpretative analysis of Marta's answers to the questionnaire that focuses on her usage and interpretation of the given semiotic means. We apply Duval's notions of treatment and conversion as described above as a lens to look at the data. The analysis intends to understand how Marta made sense of the given means. Thus, it reconstructs a possible and plausible way in which Marta may have interpreted the given and her introduced semiotic means. We then discuss Marta’s answers in terms of Rotman’s Idea, Symbol, and Procedure, with a special attention to the last one for its strict connection to the use of artifacts. As an *a priori* analysis, we claim that –when the student performs as an *Agent*, he carries out the task by means of symbolic operations and transformations, but the actions have poor spatial-temporal meaning, which is conveyed by the natural language in the domain of the *Idea*, as well as poor probabilistic significance, given by the domain of *Symbol*. In a sense, we can trace this tendency back to the empirical vision of stochastical knowledge, as it argued in Batanero, Godino & Roa (2004). When the student acts mainly as a *Person*, he intuits the general sense of the activity, but he has neither the ability to perform operations, nor the possibility to access the inter-subjective meaning of it in the realm of probability. He may intuit the notion of change, the variability of cases, the likelihood of outcomes, but he hardly manipulates the representation he had been given (or he produces by himself). Given its relation to the mathematical theory, it can be part of a formal vision of stochastic thinking (Batanero, Godino & Roa, 2004). When the student is a *Subject* that discards the

Person and the *Agent,* he can access the inter-subjective, cultural meaning of the activity, but he hardly performs actions and refers to the concrete artifact.

In this paper, we thence observe how a preservice teacher acts, namely whether she acts as Person, Subject, Agent, or whether the various agencies are somehow integrated. Our claim is that the passage from the domain of Idea to the one of Symbol and to the Procedure, as well as the link among them, needs to be supported.

Analysis

Marta's answers to question 1 and 2 can be seen in figure 2.



Immaginiamo di giocare con due ruote colorate, come queste:

Esse ruotano attorno a un perno, posto al centro. Vengono messe in moto e dopo alcuni giri si fermano. In basso un puntatore ci permette di dire se sia uscito il rosso o il verde nella ruota di sinistra, il giallo o l'azzurro nella ruota di destra.

Domanda 1

Se lanciamo ciascuna ruota 20 volte, quali saranno le possibili uscite? Motiva le tue risposte.

Ruota rosso-verde:

58,3 : 20 = 41,6 : 20

Ruota giallo-azzurro:

16,6 % G 83,4 % A ~~16,6 : 83,4 =~~

100 - 16,6 = 83,4 Maggior probabilità di uscita dell'azzurro

Domanda 2

4 studenti hanno provato a girare la ruota giallo-azzurro per 20 volte, registrando ogni volta l'uscita su una tabella orizzontale. Ecco le uscite:

Come commenteresti i risultati?

Figure 2: Marta's responses to question 1 and 2.

For the first task we observe that she changed the pictures of the spinners by including markers (dotted lines) and she wrote symbolic expressions for both spinners. For the second spinner she also wrote a sentence. In order to reconstruct what Marta was possibly thinking we have to interpret her writings and possible relations between them. First, Marta performs a congruent conversion from the pictures of the spinners to the symbolic system of percentages which is supported by the marker lines. (The conversion is congruent since the components of the spinner representation (area sizes with the labels of the colors) are mapped in a 1-1 manner into the percentage system. The order structure remains in the percentage system. Since it is not easy to determine the proportions expressed by percentages she may have used halfs, quarters or sixths of the spinner. It is not clear whether she estimated/calculated the percentages or simply used a calculator (since there is no documentation of how she arrived at the percentages compared to her detailed writings elsewhere). The interpretation of coordination is also supported by the writing of “G” (yellow) and “A” (blue) next to the percentages “16,6%” and “83,4%”. The “16,6%” is obtained or justified by the treatment in the symbolic system of percentages – the calculation “100-16,6=83,4”. Here, Marta used the complement in the percentage system. Besides this coordination Marta's way of searching for an answer differs for both spinner. For the first spinner, the writing “58,3:20=41,6:20” supports the interpretation that Marta wanted to relate the proportions to the 20 trials (“:”) and to compare the two expressions. Here, the “=” sign is used in an unconventional way. It can be seen as a statement about the action (comparison) that has to be carried out and not as the result of a comparison.

Marta then performed a congruent conversion from the system of percentages to the system of fractions (“58,3/20=41,6/20”) and started a treatment - canceling - that she did not finish. For this spinner she did not write anything else. Why did Marta stop here? One reason could be that the “=” sign takes on a different meaning now and Marta realizes that both sides are not equal. Another possibility is that she realizes that her calculations did not lead her to an answer to the question. For the second spinner Marta proceeded in a different way. She did not relate the proportions expressed by percentages to the 20 trials but she tried to compare them directly. This is supported by her writing “16,6:83,4=” which could be understood as an action that has to be carried out and the congruent conversion to the language system “Greater probability for the outcome of blue” (translation). For the second task, Marta again does not work in a consistent way. For the first list, she performs a congruent conversion from the list to the symbolic system of percentages (equal frequency of yellow and blue → “50 - 50”) and then to the symbolic system of fractions (“50 – 50” → “10/20”) where she performs a treatment (canceling: “10/20” → “1/2”) and then converts this congruently to some idiosyncratic symbolic system of letters (“1/2” → “G=A”). For the other three lists, she performs congruent conversions from the lists to the symbolic system of fractions regarding the frequencies of yellow and then to her letter system. For the second list, no treatment (canceling) is possible (“3/20”). She performs this for the third list (“14/20” → “7/10”) but not for the last one (“4/20”). What do the unconventional writings in the letter system stand for? Possible interpretations are that they describe the relation of the frequencies of yellow and blue, or represent frequencies that for Marta could be seen as the same as probabilities or as estimates of the relation of probabilities from the frequencies that may refer to different spinners with different proportions. It becomes not clear at this point that Marta is aware of the difference between frequencies and probabilities and if she knows about the relations of them. To know this difference, in fact, would mean to be able to perform non-congruent conversions between represented probabilities and represented frequencies. As it has also been pointed out by Batanero, Godino & Roa (2004), the classical and the frequentist approach to probability are complementary in nature, in that in random experiments at each trial different results are obtained, namely –unlike in arithmetic or geometry– experiments cannot be reversed, and it is only by means of combinatorial schemes that the students make sense of probabilistic problem. When we look at the writing left of the lists the first interpretation gets supported since Marta added the absolute frequencies for yellow from all lists, related them to 80 trials, and connected this to the letter system ("→G"). In her writing, the "69" for blue may be a result of a reference to "100%" as the whole and the usage of the complement idea. There are no further comments or interpretations given by Marta.

The inconsistencies of Marta's answers may be an indicator for her trying to make sense of the given new situation, and particularly the given means. The dominance of symbolic expressions may be due to Marta's experience from other domains in mathematics. Marta related percentages, proportions of the spinner and fractions to each other and the word "probability" by performing *congruent* conversions and she performed treatments in the systems of percentages and fractions. We can conclude that Marta used her knowledge about the symbolic systems of fractions and percentages that she gained in other domains of mathematics as arithmetic. However, there is no comment from her on the possible outcomes and there is no interpretation of "probability". The idea of variability also is not present in her writings. The intended meaning is accessed neither in spatial-temporal terms nor in cultural and symbolic terms. However, Marta's answer to question 3 shows a leap: "I expect a majority of blue" ("Mi aspetto una maggioranza di azzuro."). Here, she might start to access the cultural form of reasoning about the likelihood of a sequence of outcomes: she uses the verb "to expect", which is historically related to the mathematical expectation, and the term "majority", which is related to proportions. However, there is no explicit justification as a reference to the spinner and it is not clear, whether these terms have the conventional meaning for Marta. Moreover, we did not find any evidence in Marta's written answers that she has used any out-of-school experience with random situations.

BACK TO IDEAS, SYMBOLS AND PROCEDURES

Starting with Marta's case, we reflect on stochastic thinking in Rotman's terms of *idea*, *symbol* and *procedure*. If a student like Marta relies mainly on procedures, then there is only a weak or pointless reference to Idea and Symbol. The student carries out the task by means of symbolic operations and transformations. The actions have poor spatial-temporal meaning, which is conveyed by the natural language in the domain of the Idea, as well as poor probabilistic significance, given by the domain of Symbol. Sometimes it is possible to carry out a procedure without accessing the meaning and significance of it. This occurs for example when the student simply manipulates signs or may occur when already known systems of symbols from different domains of mathematics are used which may lead to *substituting strategies* in solving probability problems (see also Stanja, 2012). This risk can be felt as very cogent, if we consider that the students generally tend to import their beliefs about the deterministic nature of mathematics into probability (Meletiou-Mavroteris & Stylianou, 2003). When considering the conversions that Marta did, we noticed that she performed only congruent conversions (1-1-mappings). This would be not troublesome, when task would be to determine fractions or percentages from circular representations. In the frame of stochastic this is not sufficient: the interpretation and usage of the given means is more sophisticated in stochastic. In answering the tasks in the questionnaire, the students not only had to coordinate the various means, but it would have been useful to shift from the spatial-temporal and sensorimotor domain, from the intuitive world of ideas, to the culturally-given and socially-shaped world of the symbols, where the spinner changes from a real world object to a mathematical sign referring to probability. In fact, there is an underlying intended meaning when the students are asked to comment a given sequence of outcomes: the notion of likelihood, a historical concept that had been developed in order to model the extent to which experimental evidence (from one or many trials) supports the expected "ideational" outcome. The intended meaning of the activity is to shape a shift from the artifact of the spinner towards the signs embedded into the table. When the students are asked to predict an outcome at their turn, another underlying intended meaning is present: the notion of variability. Variability can be connected in this case to the notion of Model Fit (Konold & Kazack, 2008). We refer the reader to Prodromou (2011) for an insightful study on the students' ways of relating experimental and theoretical perspectives on probability. From Prodromou's research it emerges that students have difficulties in coordinating the actual data and the ideational model.

CONCLUSIONS AND DISCUSSION

This contribution is a first step to explore Rotman's frame. Despite the necessity of further investigating whether and how this frame serves to be useful in studying stochastic thinking, in the present study we focused on the procedures on semiotic means. The novelty of this contribution consists in giving insights on Rotman's frame, viewed in terms of Duval's semiotic. As a first step, we focus on a task involving randomness. Following Duval (2008), the ability to perform conversions are a crucial point in learning processes. Marta's example illustrates that the means used in stochastic are cultural products used by a stochastic community and whose usage is not self evident and self explaining. The inconsistencies in her usage of the means indicate that she did not answer the questions only with some pre-existing ideas in mind but that she was trying to make sense of the situation in front of her while interacting with it. It became clear that also the meaning and usage of known means (from arithmetics) need to be changed: it should be addressed deliberately and explicitly and therefore should be part of stochastic teaching. To our knowledge, this is not the case at the present, and there is a need for the design of suitable learning activities. Stochastic as cultural product implicates a different perspective on learning/the contents to be learned. Speaking in Rotman's terms this would mean that the links between the domains of Idea, Symbol and Procedure need to be supported. The meaning of interaction with cultural artifacts (semiotic means) and with other persons is emphasized. Related to this is teachers' training. Following Batanero, Godino & Roa (2004), and with Rotman (2003), we maintain the importance for the teachers to integrate theoretical and empirical experiences in order to make sense of the task, fostering the Person, the Subject, and the Agent in a coherent whole.

A last point considers methodological issues. The types of tasks used did not allow doing only procedures, and they do not ask for doing only procedures as well. They differ from tasks that ask for solving a problem by performing procedures. Following Sierpinska (1994), we distinguish between the solving of problems and understanding. However, the method of questionnaire showed its limits in studying stochastic thinking when understood as a process. To get a deeper understanding of the (evolving of) stochastic thinking, other methods – as interviews – which allow us to trace the processes need to be considered.

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