(DIS)ORIENTATION AND SPATIAL SENSE: TOPOLOGICAL THINKING IN THE MIDDLE GRADES

Elizabeth de Freitas & MaryJean McCarthy

Adelphi University

In this paper, we focus on topological approaches to space, and we argue that experiences with topology allow middle school students to develop a more robust understanding of orientation and dimension. We frame our argument in terms of the phenomenological literature on perception and corporeal space. We discuss findings from a quasi-experimental study engaging 9 grades 5-8 students in a 6-week series of school-based workshops focused on knot theory. We discuss video data that shows how students engage with the intrinsic disorientation of mathematical knots through the use of gesture and movement.

Spatial sense, topology, corporeal space, embodiment, knots

INTRODUCTION

Fielker (2011) suggests that we need to broaden our conception of geometry and recognize that "geometries" are diverse kinds of approaches to space, some more suitable than others to the study of movement, transformation, connectedness, dimensionality and orientation. Taking this more expansive approach by inviting students to experiment with spatial reasoning creates an opportunity for students to attend to the corporeal and material aspects of mathematics. We focus on topological approaches to space, and we frame our analysis in terms of the literature on perception and corporeal space. We discuss data that suggests students' gestures often function as devices for orienting a knot in relation to a moving frame of reference. Our analysis suggests that the students are operationalizing gesture and vision and voice to literally dislocate themselves in relation to the knot so as to better explore it, and that the experience provides an opportunity for them to develop the topological concept of dimensionality, which we define as degrees of freedom of movement.

METHODS & DATA

Nine students (grades 5-8), three female and six male, were recruited from a New York State elementary school to participate in six two hour afterschool topology workshops over a three month period. These workshops led by the two principal investigators were offered as non-credit extra-curricular opportunities. The participants had not studied topology previously and their school math curriculum had not provided many opportunities for exploring spatial reasoning. The following questions guided our research: How do students solve problems that entail topological approaches to space? In what ways do gestural-haptic modalities factor into students' spatial reasoning as they engage in problems through topological rather than Euclidean concepts? How does the concept of dimension factor into students' spatial reasoning, and in what ways can problem solving with knots and knot diagrams develop a robust topological concept of dimension? We focused on knot theoretic activities that involved identifying, creating, modifying, comparing, sorting, decomposing and diagramming knots. Various media were used, such as strings, ropes, pipe cleaners, sculpie clay, ipads, and paper and pencils. The software Knot Plots generated dynamic images of knots which could be spun around and modified by the students. The focus of the activities was on how students moved back and forth between image, object and diagram, and how this entailed particular working concepts of dimensionality, directionality and orientation. Although the concepts of invariance and non-rigid transformations were introduced and explored at the first workshop, the PIs gave no other direct instruction, allowing students to develop collaboratively various knot theoretic tools as they problem solved, such as the (un)crossing number, the Reidemeister moves and the colorability constraints. Activities that shed significant light on the questions guiding the research were tasks that were: (1) open-ended and invited inventive diagramming practices for representing 3-D objects, (2) involved working with orientable and non-orientable surfaces, (3) entailed identifying discrete Reidemeister moves in continuously unravelling mathematical knots displayed in video. Project data consists of video and audio recordings of (a) workshops and (b) performance-based interview tasks completed 6 weeks and 6 months after completion of the workshops, as well as (c) drawn artifacts produced by students during both workshops and interviews. In order to focus carefully on the students' entire corporeal space as they worked, we coded the video data by tracking (a) student use of spatial language to describe their activity, and (b) hand gesture as a form of embodied orientation (in relation to a set of coordinate axes).

THE VESTIBULAR LINE & THE VISUAL LINE

Recent research on the kinesthetic perception of physical space and students' deployment of gestural/haptic modalities in problem solving and other mathematical activity has begun to shed light on the complex facets of spatial sense (Nemirovsky & Ferrera, 2009; Núñez, Edwards, Matos, 1999). Hostetter and Alibali (2008), for instance, draw on Gibson (1979) and others to argue that perception and action are mutually determining, and that knowledge emerges through these co-adaptive processes. This "tight coupling of motor and perceptual processes" underscores the ways in which we activate sensorimotor processes when working with concepts (Hostetter and Alibali, 2008, p. 497).

The mathematician Bernard Teissier describes "mathematical intuition" as being a fusion of two modes of perception, the *visual continuum* and the *continuum of motion*. Drawing on recent work in neurophyisiology (i.e Berthoz, 2005), Teissier (2011) puts forth what he calls the "Poincaré-Berthoz isomorphism" that links these two modes, suggesting that their fusion is at the source of mathematical invention "For instance, when we perceive a mathematical line or curve, we actually perceive two fundamentally different things: One, a "vestibular line" that is dynamic, seemingly flowing, "parametrized by time and Rythmed by the steps" (p.237) and two, a "visual line" associated more with boundaries and ambient spatial coordinates. In the case of the *straight* line, we either perceive it in terms of its intrinsic mobility (constant velocity) or as a "curve having everywhere the same orientation" in relation to a frame of reference (p. 238). The cognitive research of Berthoz (2005) suggests that the human mind makes strong links between the vestibular line and visual line, which accords with Poincaré's insights, noted by Teissier, that "the position of an object in space is related to the set of muscular tensions corresponding to the movement we must make to capture it by the equivalent of a coordinate change" (p. 238). The fusion of these two perceptions generates a "protomathematical object" which then lends itself to all sorts of mathematizing.

In order to study the fusion of these two modes of perception, we focus here on topology defined as the study of those properties of geometric objects that remain unchanged under bi-uniform and bi-continuous transformations (Debnath, 2010). Informally referred to as "rubber sheet geometry", topology is concerned with bending, stretching, twisting, or compressing elastic objects (O'Shea, 2007; Richeson, 2008; Stahl, 2005). Despite thousands of years of studying the metric relationships of polyhedra, no one prior to Euler had studied the non-metric relationships of connectedness.[1] Through the work of Gauss, Klein, Riemann and Poincaré, topology became the qualitative study of surfaces, manifolds, boundary relationships and curvature. Topology shifts our attention away from concepts of measure and rigid transformation, and focuses on the stretching and distortion of continuously connected lines and regions. In *Geometry and the Imagination* (1932/1952), David Hilbert claimed that "in topology we are concerned with geometrical facts that do not even involve concepts of straight line or plane but

[1] We know that Liebniz was already familiar with the formula, and historians speculate that Descartes was aware of a similar one (Richeson, 2008). Although Euler's solution to the bridges of Konigsberg problem (1736) comes prior to his letter to Goldbach about polyhedra, the significance of the latter is noted here due to the way it breaks with prior mathematical treatments of polyhredra.

only the continuous connectiveness between points of a figure". In this paper, we are concerned with how particular aspects of topological thinking allow for a better fusion of the two kinds of perception mentioned above. For instance, topology deploys a more robust concept of dimension than what we normally find in the geometry curriculum –rather than the 'size' of a space, topologists refer to dimension in terms of degrees of freedom of movement. Also, the concept of orientation in topology is defined in terms of the capacity to move around space and return to a particular point "oriented" in the same direction as when the motion began, rather than in terms of a fixed Euclidean frame of reference. Finally, the topological focus on distortion and stretching more generally fuses the two kinds of perception, since it transforms the static line into a mobile continuously varying object.

According to Smith (2006), "Euclidean geometry defines the essence of the line in purely static terms that eliminate any reference to the curvilinear ('a line which lies evenly with the points on itself)." He contrasts this rectilinear concept with the "operative geometry" of Archimedes, in which the straight line was characterized dynamically as 'the shortest distance between two points.'" (p. 148). Smith suggests this definition marks the line as a continuous operation and a "process of alignment" pursuing its own inherent variability.[2] A more elastic definition of the line attributed to Heron is "A straight line is a line stretched to the utmost" (Metrica, 4. Gray, 1979, p.128).

Student experiences with knots afford opportunities for developing an understanding of orientation and dimension as operative concepts rather than attributes. Instead of defining the line solely in terms of Euclidean measure and planar existence, knots embody a twisting stretching multi-dimensional line of flight that breaks through the plane. Knot theory emerged in the eighteenth and nineteenth century and developed within the field of topology. Mathematical knots are closed multi-dimensional (knotted) curves that are deemed equivalent if one can be deformed into another (note that this deformation is not a Euclidean rigid transformation). Mathematical knots take up the line as a non-linear and non-planar process of becoming (without end or origin), a process of actualization whereby new dimensions and new entanglements unfold. The curve or line in n-space is a multi-dimensional entity, suddenly possessing perspective and depth. One can relax the knot and loosen its crossings, and then imagine crawling along the rope, following its path into the depth of the page. The knot has no interior or exterior; it is all line, or all outside. Recent interest in including knot theory in the middle school curriculum points to how it helps students develop spatial reasoning

[2] According to Netz & Noel (2007), Archimedes treated diagrams as physical models while attending more to the "broader, topological features of a geometrical object" (Netz & Noel, 2007, p. 105).

through visual and tactile exploration of both quantitative and qualitative invariants (Adams, 2004; Handa & Mattman, 2008). Knot theory offers students a creative and generative approach to 'spatial reasoning'. It captures the dynamic multiplicity of space as a process of dimensional unfolding. One is always in the middle of a knot, pursuing its lines of flight.

KNOTS AND KNOT DIAGRAMS

From 1988-1991, Carol Strohecker ran a "knot lab" in an urban elementary school where 20 fifth graders explored topological thinking through the study of knots. Students used string and other media to compose knots, developing and discussing strategies for doing so. Using the snake method, which involved placing the string on the table, identifying a starting point and fixing one end of the string while moving the other end, students were able to perceive the relationships between strands differently, conceptually integrating the more entangled parts with the various loops that fed into them. The use of this method also seemed to occasion students' efforts at re-orienting their knots (rotating them on the desk), which further developed their "body syntonicity" (Papert, 1980) in that they began to decenter themselves as a perceiver located at a fixed position, and instead identified with the mobility and disorientation embodied in the knot. Strohecker suggests that the medium itself (the pliable string) afforded students an opportunity to develop their spatial reasoning in this way, describing how "many of the children involved their bodies in expressing their conceptions of knots and knot-tying, often relying on their arms or legs to represent ends of string moving into the form of a knot." (p.6). She also indicates that student language use while describing their knots, for instance expressions such as "up/down, top/bottom, above/below, over/under, in/out", revealed how they worked the string as though it were a boundary, dividing space into neighbourhoods that were either in or outside of the knot. Her research clearly shows that students dealt explicitly with topological concepts.

As Kuechler (2001) suggests, the capacity of the knot "to fashion decentred spatial cognition" (p.82) explains in part our fascination with knots in textiles and symbolic forms. She offers an "ethnography of knots" pointing to the prevalence and power of knotted effigies and knotted patterns across various cultures and times. Knots seem to refuse to be seen from one particular point of view or perspective. Knots are all movement along a curvilinear line, evoking fluid spatial relationships. "Each knot is, in a sense, its own universe, which invites contemplation of its topology both as it is being formed and as a completed object" (Strohecker, 1991, p. 215).

For middle school students, one of the biggest challenges – as well as being one of the richest areas for developing spatial reasoning – involves tasks of creating and decoding *diagrams* of knots. Knot diagrams introduce depth into the plane, conjuring a virtual dimension within the two-dimensional surface. The crossings in knot diagrams create a multi-dimensional effect, suggesting a layering precisely where Cartesian geometry would have imposed an intersection. Making sense of knot diagrams demands that one construe an "over/under" relationship in two dimensions and that one follow the continuity of a line as it seems to leap off the surface of the page. Moreover, knots are defiantly without orientation, and yet diagrams are attempts to capture and orient knots on the plane. Students have to decide what is in the foreground/background (and in some cases how that relationship might be evoked) and decide on a perspective and an orientation. Because of the absence of axes and other straight lines to structure one's vision, and because students are learning to pay attention to topological relationships rather than Euclidean ones, drawing knot diagrams often entails positioning oneself (as the observer) in multiple and diverse locations. Châtelet (2006) argues that knots and knot diagrams disrupt fundamental Euclidean spatial practices. They introduce a new manner of intervention and a new way of making mathematical images. They express both entanglement and rupture in the way they disobey the plane. Thus the knot and its diagram contest the usual epistemological barrier between geometric space and corporeal space. Thus any attempt to locate the knot within a mathematical frame of reference is complicated by its 'tied' nature, its folds and twists, connectedness and relationality.

ORIENTATION, MOVEMENT AND CORPOREAL SPACE

We see from the literature that (dis)orientation and dimensionality figures prominently in our engagements with knots. Ahmed (2010) argues that orientation is an essential aspect of spatial sense making. Put simply, "orientations are about the direction we take that puts some things and not others in our reach." (Ahmed, 2010, p. 245). Body orientations thus shape and map space by generating operative axes around which we define our movements. Thus orientation marks a "here" and a "now" from which we proceed. One might even suggest that orientation and motion are mutually implicated. With reference to the phenomenology of Husserl, Ahmed (2010) suggests that orientation marks a "zero-point" or starting point "from which the world unfolds" (p. 236). This implicit structuring of a zero-point for the body entails a sense of movement or potential movement, and this in turn conditions our perception of what is foreground and what is background. Indeed, the perception of depth (and dimensionality) is not as simple as one might first imagine, as it depends on a tactile-kinesthetic fusion of sensory impressions. The body is thus "something that I move with, not something I move, i.e., it has the characteristic of direct motility – I do not have to place my body in order to move it." (Rush, 2008, p. 18).

CASE STUDY

In this section, we discuss data collected during the post-intervention interviews with the student participants of our study. We focus on one task where the students were shown a 2-D depiction of a knot (figure #1), asked to explain if and how they might simplify the knot, and then asked to identify key moves in a video of the same knot unravelling into the unknot. We discuss below how Maya moves back and forth between the vestibular line and the visual line as she engages with this picture of a knot. The image demands a great deal of depth perception since there are loops under loops under loops. This layering of the rope makes for a complex perceptual task. Each time one focuses on one crossing and 'sees' a particular relationship of over/under on a plane of reference, one is then forced to dislocate that plane of reference when either the same two strands reverse the relationship of over/under at the adjacent crossing or a third strand appears beneath the other two and the student has to penetrate the imagined plane to incorporate this third strand in making spatial sense of the relationships. Despite the seeming complexity of the knot, it is reducible to the unknot after a series of moves.

Figure #1

Figure #2

Maya is first asked to count the crossings. She tilts her head and body as her fingers trace the path of the knot (figure #2). She names the crossings as either over or under *because* her finger follows the vestibular line and thus there is a *relative* experience of over or under. Then she stops and looks up and says, "but if you look at it from the other perspective wouldn't it be over or under?" (0:15) When asked to say more, she removes her hands, and explains, "You can see this one going under, this one going under, but you have to like focus on one and the one that you focus would go over or under. (0:52)

In so doing, Maya is shifting from perceiving the vestibular line to perceiving the visual line. With the visual line, the concepts of over and under cannot be assigned to a crossing unambiguously. In other words, at each crossing there are two strands, so there is no sense that a *crossing* has a definitive over or under designation. Such a designation only makes sense if one imagines oneself actually moving along one of the strands on the vestibular line.

Asked where she would start in order to simplify the knot, she picks a strand (lower right for her, labelled A on figure #3 below) and gestures as though she had used her pointer finger to stretch and pin the strand (Figure #4). She affixes this finger to the page, as though she were holding down the rope, while her other hand takes on the gesture of a pincher or grabber, hovering and rocking slightly back and forth in the air above the knot.

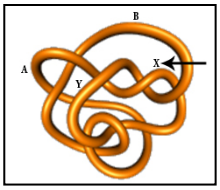


Figure #3

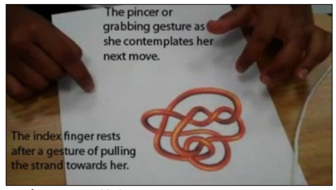


Figure #4

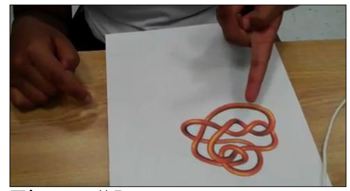


Figure #5

She then places both hands on the table and she taps her fingers rhythmically and in unison while she thinks. She switches her first answer and picks a second strand (labelled X on figure #3), and then she gestures to pull and stretch the strand out from under strand B. She flips her hand palm up (figure #5) to show that it would "go like that". This is not a pointing gesture, even though it looks like one. It is a flipping gesture that is meant to embody the inversion required as the loop moves from background to foreground. The hand is working as a proxy here as she embodies the new sense of orientation. She could have kept her hand oriented as in figure #3 and simply indicated the grabbing or pincering of the strand and its being stretched to a new location, but in order to capture the new relative relationships between over and under, she inverts her hand so that it's palm up. This is significant. It shows how she is following the vestibular line using the orientation of her hands (palms up or down or other). What was under is now over, what was up is now down – she is back to following the vestibular line and enacting – through her hands – the changing relationships. In this sense, the hands are proxies for the disorientation embodied in the knot. In relation to the space of the room, Maya is upside down and looking up at the backside of the knot, as though she were on the other side of the paper. Maya is shifting her projected perspective on the knot as she engages with the task. In other words, she is moving around the knot – within, behind, beside, on top – in ways that speak to her embodied engagement. It is usually her left hand that performs the flipping while the right hand performs the stretching. She then pauses saying "Oh, this one's complicated" and spins the sheet of paper around, until it is oriented as in Figure #6.

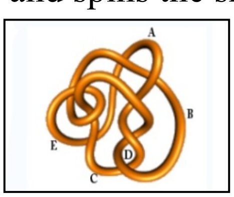
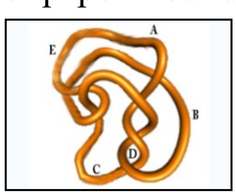
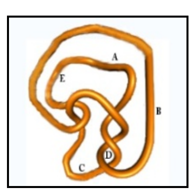


Figure #6



Figure#7 First Move



Figure#8 Next Move

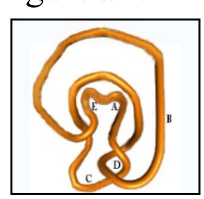


Figure #9 Next Move

Rotating the paper shows that she is engaged with the disorientation of the knot and that she is aware that the image isn't locked into a particular perceptual grid of 'proper' up/down orientation. She then confidently suggests a series of moves: pulling strand C out, flipping strand C over D and towards B, pulling strand A out, and then flipping strand E over the body of the knot. Although the first two moves don't seem to simplify the knot, the combination of these last two moves is indeed a move that will begin the unravelling (notice that strand E is linked to strand A in such a way that the two are part of a loop that is buried beneath the knot, and three of the adjacent crossings are under the rest of the knot). One can grab and stretch the loop towards the left, eliminating these three crossings (Figures #7).

One of the challenges in this task is 'seeing' into and behind the knot, shifting one's imagined perspective, and noticing these kinds of patterns related to depth and adjacency. This entails shrinking and stretching lines, which we found to be associated with student gesturing of both hands simultaneously. One hand was consistently used to gesture the pulling and stretching of a particular strand, but both hands were used – as though at either end of an elastic – when the students wanted to eliminate a crossing using Redeimester moves #2 and #3.

CONCLUSION

Studying students' experiences with topology revealed how orientation is a complex component of spatial sense. Analysis of video data showed how students' corporeal space entailed a moving rather than fixed perspective, and that gestures embodied this implicit mobility. Rather than seeing the gestures as iconic or indexical, we analyze them as embodiments of perspective. Students' gestures reveal how they follow both the vestibular line and the visual line as they make sense of knots. We see here how the two modes of perception – the visual continuum and motion continuum – were taken up in her gestures as she pursued the shifting orientation entailed in the diagram.

REFERENCES

Adams, C. (2004). Why knot? An introduction to the Mathematical Theory of Knots. Englewood, Colorado. Key College Publishing.

Ahmed, S. (2010). Orientations matter. In Coole, D. & Frost, S. (Eds.). (2010). New materialisms: Ontology, agency, and politics. London: Duke University Press. 234-257.

Châtelet, G. (2006). Interlacing the singularity, the diagram and the metaphor. In M. Duffy (Ed.). Virtual mathematics: The logic of difference. Clinamen Press. 31-45.

Debnath, L. (2010). A brief historical introduction to Euler's formula for polyhedra, topology, graph theory and networks. International Journal of Mathematical Education in Science and Technology, Vol. 41, No. 6, 15 September 2010, 769–785.

Fielker, D. (2011). Some thoughts about geometries. Mathematics in schools. 23-26.

Gray, J. (1979). Ideas of space: Euclidean, non-Euclidean and Relativistic. Oxford: Clarendon Press.

Handa, Y. & Mattman, T. (2008). Knot theory with young children. Mathematics Teaching. 211. 32-35.

Hostetter, A. B., & Alibali, M. W. (2008). Visible embodiment: Gestures as simulated action. Psychonomic Bulletin & Review, 15, 495–514.

Kuechler, S (2001) Why Knot? Towards a Theory of Art and Mathematics. In: Pinney, C and Thomas, N, (eds.) Beyond Aesthetics: Art and the Technologies of Enchantment. (57 - 77). Berg: Oxford.

Nemirovsky, R. and F. Ferrara (2009). Mathematical Imagination and Embodied Cognition. Educational Studies of Mathematics, 70(2), 159-174.

Núñez, R., Edwards, L. & Matos, J.F., (1999). Embodied cognition as grounding for situatedness and context in mathematics education, Educational Studies in Mathematics, (39), pp. 45-65

O'Shea, D. (2007). The Poincaré conjecture: In search of the shape of the universe. New York: Walter and Company.

Richeson, D. S. (2008). Euler's gem: The polyhedron formula and the birth of topology. Princeton, NJ: Princeton University Press.

Rush, F. (2009). On Architecture. New York: Routledge.

Smith, D.W. (2006). Axiomatics and problematics as two modes of formalisation: Deleuze's epistemology of mathematics. In. S. Duffy Virtual mathematics: The logic of difference. 145-168.

Strohecker, C. (1991). Elucidating styles of thinking about topology through thinking about knots. In I. Harel & S. Papert (Eds.), Constructionism. Norwood, NJ: Ablex. 215-234.

Teissier, B. (2011). Mathematics and narrative: Why are stories and proofs interesting? In A. Doxiadis & B. Mazur (Eds.). Circles disturbed: The interplay of mathematics and narrative. Princeton University Press. 232-243.