TEACHERS' GEOMETRICAL PARADIGMS AS HIDDEN VARIABLES IN THE CONTEXT OF MATHEMATICAL WORLDVIEWS AND GOALS OF EDUCATION

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This article presents some results of a qualitative study on secondary teachers' beliefs, reconstructed as so-called individual curricula, a concept to represent a teacher's argumentative connections between his choice of content, methods, and goals of education. Within these individual curricula, two archetypes are figured out that are supposed to be oppositional in three dimensions: in the use of Geometrical Working Spaces in classroom teaching, in the general mathematical worldview, and in the choice of goals of education a teacher intends to achieve by teaching elementary geometry. The first archetype is characterised by deductive standards, a static view on mathematics, and expert-oriented goals of education; the second one is more empirical, dynamic, and guided by pragmatic goals of education.

INTEREST OF RESEARCH AND BACKGROUND OF THE STUDY

This article presents some results of a qualitative interview study concerning secondary school teachers' individual curricula on teaching elementary geometry. The core framework is based on the concept of *individual curricula* (Eichler, 2007) which are used to describe the part of a teacher's *beliefs system* (cf. Philipp, 2007) that contains argumentative connections between content, methods, and goals of education and has a similar function as a written curriculum (cf. Stein, Remillard & Smith, 2007), especially the task to justify the choice of contents and teaching methods against to the background of a teacher's individual goals of education.

After reconstructing nine individual curricula out of in-depth interviews, the study was faced to the problem to compare and to categorise the findings. Since an individual curriculum – even restricted to teaching elementary geometry – is a "holistic" conception, it is not sufficient to use just one framework for a categorisation, e. g. just a purely geometrical one; rather it is advisable to use discriminations on three typical levels of a curriculum: the level of goals of educations, the geometrical level, and the geometrical aspect seen in a broader context of general beliefs of the "nature" of mathematics. To do so, three background theories were combined, namely the *theory of Geometrical Working Spaces* (Kuzniak, 2006), a classification of *goals of education* (Graumann, 1993) and a framework to analyse general understandings of mathematics, called the theory of *mathematical worldviews* (Grigutsch, Raatz, & Törner, 1998). Insofar, the central research question of this study is as follows: How can individual curricula on teaching elementary geometry be classified based on these three levels and are there any systematic connections between them? It will be argued that the answer is positive and that it is possible to identify two archetypes of systematic connections between these levels and that each of the nine teachers can be attached to one of the two archetypes.

THEORETICAL BACKGROUND

Before we can start to describe the study and its method, it is necessary to make some remarks on the three theoretical backgrounds used for the classification.

Geometrical Working Spaces

The framework of Geometrical Working Spaces (GWS) is based on the idea that three *geometrical paradigms* are relevant to the history and philosophy of elementary geometry which are fundamentally different in ontological, epistemological, and practical assertions (Houdement & Kuzniak, 2001). The classification consists of three entries which are named and explained as follows:

1) Geometry I or G1 (Natural Geometry): Geometry is regarded as an *empirical discipline* which refers to physical objects. To "proof" or to refute conjectures, both *argumentations and experiments* are allowed. The basic foundations of arguments are not axioms, but propositions derived from empirical observations. The standards of arguments are typically not as "sophisticated" as in mathematical proofs, but close to ordinary language argumentations used in everyday life.

2) Geometry II or G2 (Natural Axiomatic Geometry): Geometry is seen as an *axiomatic theory*. The axioms are supposed to refer to the real world and, therefore, to describe physical figures and objects (with some idealisations); but to proof or to reject propositions, no empirical arguments are allowed. Only *deductive* conclusions based on the axioms are permitted.

3) Geometry III or G3 (Formalist Axiomatic Geometry): Geometry is seen as a *formal axiomatic theory*, and no connection to the real world is intended.

G3 is more or less restricted to university level, whereas G1 and G2 are the paradigms that play a role at secondary school. Against to the background of geometrical paradigms, a *pupil's* Geometrical Working Space can be described as his individual (conscious or unconscious) selection of aspects of one or more geometrical paradigms he uses when being confronted to geometrical tasks, concepts, figures, and problems (Kuzniak, 2006). This approach was extended to analyse *teachers*' standards of teaching geometry (Girnat, 2009). In this case, the teacher's GWS is not necessarily his own working space, but the working space he demands from his pupils to use. Insofar, the teacher's GWS expresses what type of geometrical paradigms he wants to see as predominant in his lessons on geometry.

According to Houdement & Kuzniak (2001), the main problem on teaching geometry consists in the fact that a written curriculum normally intended the use of G2, whereas pupils often adhere on G1. Girnat (2009) pointed out that the teachers' response to this problem is quite diverse: Some of teachers try to implement G2 standards as their intended GWS, but some prefer to teach geometry on a G1 level, partly intentionally to avoid pupils being demanded more than "appropriate", partly unintentionally since G1 is their own understanding of geometry.

Mathematical worldviews

A mathematical worldview can be explained as a *beliefs system* (cf. Philipp, 2007) which a person, especially a teacher, holds for true and "essential" in all parts of mathematics. We follow an approach of Grigutsch, Raatz, & Törner (1998) who suggest a classification of mathematical worldviews by four aspects:

1) *Formalistic aspect*: Mathematics is seen as a formalistic language whose concepts are introduced by definitions and whose theorems are derived by deduction from basic axioms.

2) *Schematic aspect*: Mathematics is seen as a pool of rules and algorithms which enables a person to solve mathematical problem by following these rules and algorithms (like recipes, i. e. not necessarily by understanding their backgrounds).

3) *Dynamic aspect*: Mathematics is seen as a field of creativity in which everyone can try to invent his own concepts and rules to solve mathematical problems or problems including a mathematical part. The opposite is called the *static aspect*, which means: Mathematics is seen as a bound of theories whose concepts, axioms, and theorems are fixed and unchangeable; and doing mathematics means reproducing these theories and to applying them correctly.

4) *Applied-oriented aspect*: Mathematics is seen as practically useful and as a powerful tool to handle challenges occurring in everyone's professional and everyday life.

Grigutsch, Raatz, & Törner (1998) undertook a representative study among secondary school teachers (N=400) to reveal correlations between the four aspect of their mathematical worldviews. Their results are presented in fig. 1.

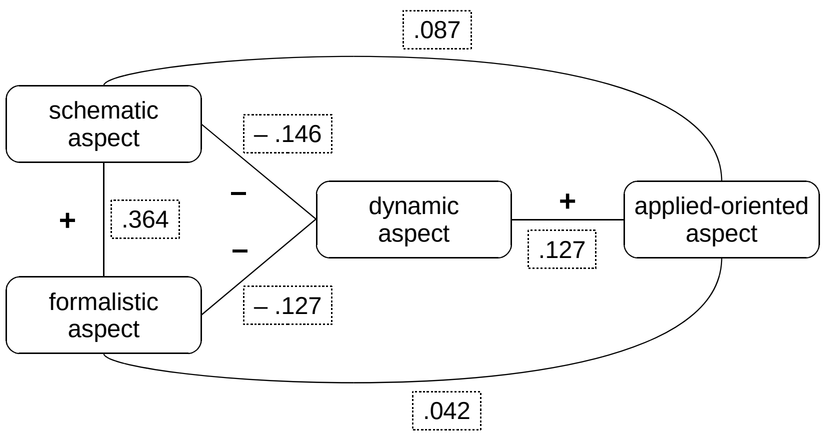

Figure 1: Correlations between the aspects of mathematical worldviews

The correlations are low at all hands, but nevertheless, Grigutsch, Raatz, and Törner conclude that there are two clusters, namely a cluster which consist of the schematic and the formalistic aspect and an opposed cluster which is formed by the dynamic and the applied-oriented one. This hypothesis is taken into account in our study. More precisely, there are two questions of interest: 1) Can similar affinities between the four mathematical worldviews observed in the study; 2) and are there connections between a teacher's mathematical worldview and his choice of the GWS he demands his pupils to use.

Goals of education

There are many approaches to classify goals of mathematics education. To analyse our teachers' statements, we choose a model that depends on two steps of discrimination: At first, we distinguish if a teacher wants to make his pupils achieve competencies which are mainly *specifically mathematical* or if he is interested in using mathematical education to acquaint his pupils with goals of education that are *more general* than mathematical ones. Let us call the first point of view *expert education*, the latter one *general education*. In case of expert education, the specific goals are given by the teacher's understanding of mathematics and are related to his mathematical worldview or (nor specially) to his geometrical paradigm. In case of general education, we use a framework of Graumann (1993) to distinguish between five dimensions of general education:

1) *Pragmatic dimension*: Mathematics education should be perceived as useful to solve practical and technical problems.

2) *Enlightenment dimension*: Mathematics education should foster an understanding of the world including its historical, cultural, and philosophical backgrounds.

3) *Social dimension*: Mathematics education should strengthen the pupils' competencies to cooperate, to communicate, and to accept responsibility.

4) *Individual dimension*: Mathematics education should enhance each pupil's own abilities and interests.

5) *Reflective dimension*: Mathematics education should sensitise the pupils to the limits, boundaries, and fallacies of mathematical methods.

The choice of this framework is founded in need of a conception which is decidedly not restricted to mathematical education (as e. g. the widespread frameworks of mathematical competencies would be), but which is in principle applicable to every school subject.

SETTINGS OF THE STUDY AND METHODOLOGICAL BACKGROUND

The study was carried out at higher-level secondary schools by interviewing nine teachers about their *individual curricula* (cf. Eichler, 2007) of teaching geometry. All these teachers studied mathematics at university on a level comparable to a master of science without or with just a minor contingent of pedagogy or didactics of mathematics. They gained their certificate necessary to be employed as secondary school teachers in a practically oriented second step of training after their studies at university. They were chosen randomly from nine different schools.

The method to reconstruct individual curricula from interview transcripts is based on a qualitative approach, called *dialogue-hermeneutics technique*, which was invented to expose argumentative relations within belief systems (Groeben & Scheele, 2000). In our case, the argumentative relations in question are mean-ends relations between contents, teaching methods, and goals of education. The methods is based on a way to represent such connections graphically: If a teachers utters a sentence like “I do a lot of problem solving to enhance my pupils’ intellectual skills”, the aim “enhancing intellectual skill” is placed in a tree diagram on a higher level and the mean “by doing a lot of problem solving” is subordinated to this aim on a lower level. After the interviewer has compiled such a diagram in the hermeneutic stage of the method, his end-product is given to the teacher to check if he can approve the interviewer’s proposal or if he insists on revising the diagram to display his arguments correctly. This is the dialogical part of the method. In fig. 2 and 3, diagrams derived by this technique are shown which are condensed to a very abstract structure.

The interviews are prepared along the principle to give as little input as possible. Therefore, the questions normally are very open like “Could you describe your lessons on geometry?” and typically followed by questions which are supposed to reveal the teacher’s aims like “Why do you prefer this content, this teaching method and so on?” or “Why do you do this and not for example this alternative?”

EMPIRICAL FINDINGS

Seven of the teachers who took part of our study can be classified as proponents of G2, two of them as exponents of G1. The latter ones are called Ernest and Henry. To quote a typical passage which can be used to classify a teacher’s GWS, we choose some statements of proponents of G2 first and some of Ernest’s and Henry’s later:

Ian: Geometry definitely is a well-ordered system, if you follow Euclid’s “Elements”. It is a prototypic example of an axiomatic theory. Unfolding this theory at school is impossible, but on a local level, it is a very important to make pupils argue precisely, to deduce from premises, to make them search for proofs or to retrace proofs at least. […] As a mathematician, I have to observe that pupils are not simply convinced by empirical observations.

Gertrude: It’s the central point of mathematics to argue logically and to show the pupils how logical chains of proofs are made.

Fredric: It is important to me that my students switch to an abstract level, practise pure geometry. In order to do so, applications, concrete figures, measuring and so on are rather obstacles than aids. These are no valid methods.

Dorothy: The beauty of mathematics is the fact that everything is logical and dignified. […] Everywhere else, there are approximations, but not in mathematics. There is everything in this status it has ideally to be in. [It is important for the pupils] to recognise that there are ideal things and objects in mathematics and that, in reality, they are similar, but not equal.

Ernest: A theory has got its place at university. […] Theoretical deliberations only make sense at school – like in my lessons –, if they are useful to solve practical problems. I mean authentic problems that come from the pupils' everyday life. Otherwise, a theory is deathlike. […] A proof is something conflicting. Normally, you prove theorems at school just because someone said that's the task mathematician have to do; and that's brainless, I think. For me, argumentation is more a social phenomenon to convince each other, to discuss a problem together, to help each other. That's the social aspect. […] It would be nice if we had a problem and everyone would propose different concepts, definitions, and we would try how far we can.

Henry: Proofs are of minor interest. The task is to make theorems plausible, e. g. by cutting out figures and laying them onto each other, and you can observe if they match each other; and we take this as a proof. […] DGS [dynamic geometry software] is a very useful tool. You can prove a lot by it. For example, in case of Thales, you can put a third point on this semi-circle and you can pull it hither and thither; and you will notice that the angle equals 90 degrees all the time. And so, you have proven that there are always 90 degrees, and you did it convincingly. […]At the end of the day, it's not necessary to be exact; it is necessary that pupils can solve their tasks. That doesn't have to be exact. It depends on the context. And later, it will be important to the pupils to solve problems. It won't be important to solve them elegantly or in the manner they have learnt at school; it will be just important that they are willing to face the problem and that they will come to a suitable solution, an estimation, an approximation anyhow.

Like in these quote, the crucial distinction between G1 and G2 is found in the role of justification and in the perception of geometrical objects of being empirical or non-empirical, "idealistic" ones. The first passages stress the importance of deductive proofs, whereas Henry and Ernest are willing to accept empirical observations as arguments. Beside that fundamental difference, you can observe some remarks on the topics we want to combine with the teachers' geometrical paradigms: 1) Ian stresses his role "as a mathematician", and Dorothy pointed out some ontological and epistemological aspects she regards as typical for mathematics. She concludes that they are "therefore" also important to her pupils. Henry, on contrary, does not accept the argument that an aspect of mathematics has to be part of mathematics education just because it is typical for mathematics as a (scientific) discipline. Besides these quotes, which can only illustrate the findings, the proponents of G2 tend to emphasise an expert education in mathematics and consider it as their task to familiarise their pupils to a scientifically oriented perception of mathematics. 2) The proponents of G1 stress the pragmatic benefits of geometry and the importance of mathematical education for the pupils' future life. They seem to be willing to adjust mathematical standards of exactness and justification to the circumstances which are given by a real world problem mathematics is a part of. Insofar, the exponents of G1 appear to be applied-oriented, whereas the proponents of G2 seem to fear confusions between mathematical and empirical standards of justification, if the practical use comes to close to geometry. 3) Especially Ernest emphasises the social dimension of education and seems to represent a more dynamic conception of mathematics which includes creating new concepts and exploring them in the contexts of realistic problems. 4) The observation that G1 proponents are willing to adjust mathematical standards of exactness to practical needs may indicate that they perceive mathematics as a "tool box" in a schematic manner. But this is unclear.

Let us regard some further excerpts of the interviews to search for connections to other aspects of goals of educations and mathematical worldviews among G2 proponents, since until now, they seem to be just focussed on expert education:

Gertrude: Besides proof abilities, problem solving is in fact the most important thing I want to convey in my lessons on geometry. […] I want activate my pupils to deliberate on their own and to overcome the habit "Now, we handle ten task using the recipe xy".

Ian: Problem solving is a sort of intelligent exercises. You have to remember former content, and you have to use it actively. Thinking, I mean, intellectual abilities are trained by problem solving; and to be successful, you have to be fit in mathematical basics, and you have to train them.

Frederick: Mathematics education is brain callisthenics. In other school subjects, you can learn something different, but in my lessons, you will do brain callisthenics. […] Proofs are important. You can't accept an assertion just because someone told you that it's true. You have to scrutinise everything.

Alan: I think, in mathematics education, pupils can learn to think, to structure, to solve problems. […] And beside this, I want to give my pupils an insight how the ancient Greek did it. They were very ambitious. They didn't just want to know how something was, they wanted to found why it was as it was. Normally, the pupils don't want to be inferior to them.

Christian: All the tasks provided at school are fabricated. I have no concern to provide a task that is fabricated. The pupils will accept it, and they can learn geometry even if the task is fabricated. It's the same thing in every school subject, and mathematics education has not to apologise for this fact. […] Knowing the basic principles precisely and drawing conclusions from these principles without calling them into question, I think that's something you can learn in mathematics education, and not in other school subjects.

In case of G1, we can primarily find the goal to show the practical side of mathematics and to prepare pupils to their further life, as sketched above. These quotes, on contrary, shall illustrate some typical statements in which proponents of G2 express goals of education that lie beyond subject-specific aspects: 1) It is noticeable that they disdain authentic real-world problems. They seem to regard realistic tasks only as tools to learn mathematics, and not as a subject interesting of its own. Insofar, they attach little value to the pragmatic dimension of education, but stress the enlightenment and reflective dimension, promoting formal and intellectual abilities. 2) Beside proofs, problem solving task seem to be the main focus of the G2 proponents. 3) For both proofs and problem solving, they seem to regard it as necessary to possess a broad and active knowledge of basic principles which are standardised and not committed to subjective creations. Insofar, a more static view of mathematics seems to be a precondition to teach geometry in the sense of G2 and to achieve goals of education that are seemingly linked to this kind of teaching.

CONCLUSIONS

Considering the few passages quoted here and the few teachers interviewed, it seems daring to draw general conclusions. But since it is one of the main tasks of qualitative studies not to make representative assertions, but to generate archetypes which can be tested representatively afterwards, we propose two archetypes that represent systematic connections between the three layers of our classification, namely goals of educations, geometrical paradigms, and mathematical worldviews.

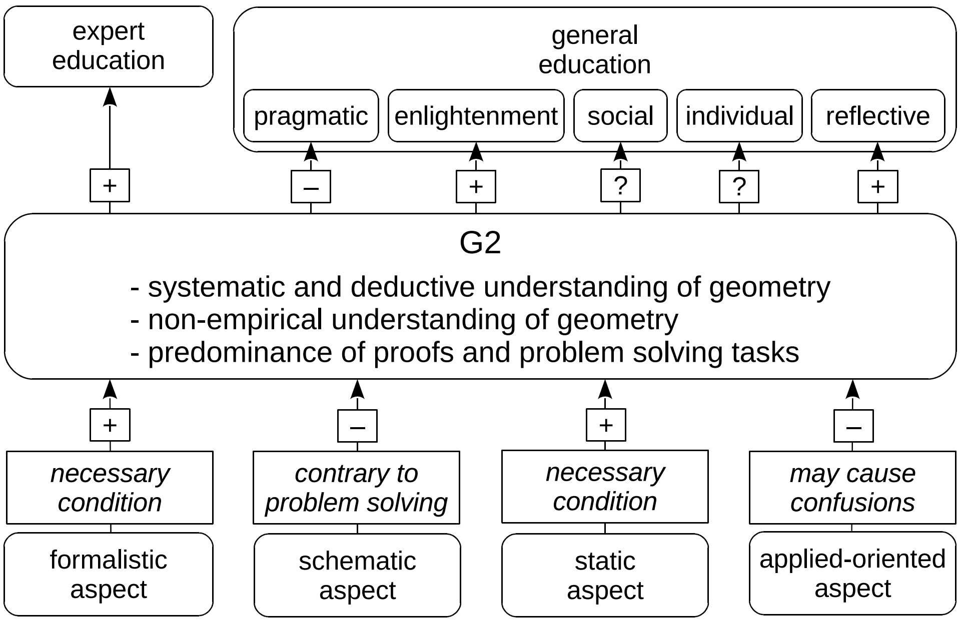

Figure 2: G2 in the context of mathematical worldview and goals of education

In fig. 2, we try to sketch the connections that are supposed to be typical for the G2 GWS. A plus sign indicates that the G2 teachers affirm a specific goal of education (at the top of the diagram) or a specific mean (at the bottom). The minus signs denote refusals. In case of a question mark, neither an affirmative nor a dismissive statement can be found. If a quantitative extension of this study was carried out, the plus and minus signs would indicate the hypotheses that a positive or rather a negative correlation should be observable. The italic phrases display typical reasons “in a nutshell” the teachers use to justify their approvals and rejections.

Overall, the formalistic and the static aspect of mathematics seem to be a precondition to implement G2 standards. Applications may be “too empirical” and could cause a conflict with the non-empirical standards of justification. It seems a “natural” way to extend the G2 approach to subject-specific goals of education, especially to an expert education and, as far as general education is concerned, to rather intellectual and cultural aspects than to pragmatic ones.

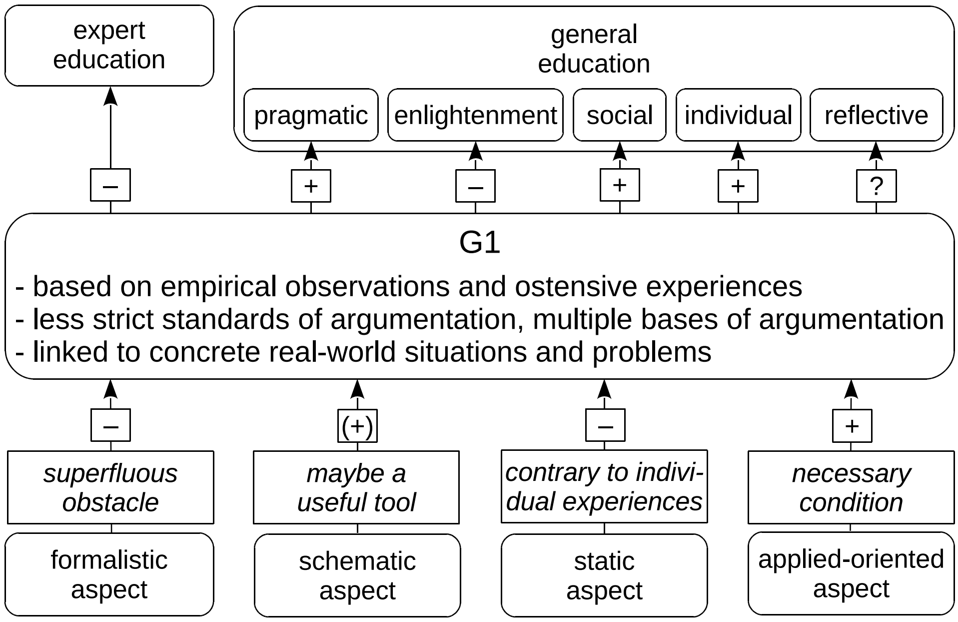

Figure 3: G1 in the context of mathematical worldview and goals of education

As visualised in fig. 3, an archetypical G1 curriculum is supposed to look quite oppositional compared to a G2 one: Realistic problems are necessary conditions for an empirical understanding of geometry, and from there, a “natural” way leads to pragmatic goals of education. Formalistic aspects are not as important as for a G2 concept and rather obstacles; and a static understanding of mathematics seems to be opposed to individual experiences.

The results of this study may be useful in two respects: 1) Qualitative studies primarily support conceptual work and hypothesis generation. In this study, we propose the two concepts of a G1 and G2 archetypical curriculum. These suggestions can be the initial points for a representative study on this issue. In this case, the two concepts GWS and “dimensions of general education” has to be operationalised); and it has to test if the connections proposed by plus and minus signs in the archetypes can be confirmed or not.

2) If these archetypes were affirmed by a representative study, it would be possible to get a deeper explanation of mathematical worldviews, since in this case, the teachers’

GWS would be a “hidden variable” that could explain systematic connections “behind” correlations as displayed in fig. 1. Maybe, it is possible to revise or to clarify some of the aspect expressed there. For example, the highest correlation observed, the one between the formalistic and the schematic aspect, is astonishing concerning the G2 teachers’ statements that they prefer problem solving and that they want to reduce the amount of schematic tasks. A possible explanation could be that they insist on routine task as a precondition of problem solving, but that they do not see schematic tasks as valuable on their own. Insofar, the connection to the archetypical curricula could be the basis to formulate the aspects of mathematical worldviews more precisely and more usefully to collect data on teachers’ curricular background of planning their lessons on geometry.

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