WHICH GEOMETRICAL WORKING SPACES FOR THE FOR THE PRIMARY SCHOOL PRESERVICE TEACHERS?

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Abstract: This paper is to describe the various GWS pre-service teachers could be working in, in connection with the different mathematics curricula implemented in primary and secondary schools in the province of Québec (Canada). It results from this study that the GWS[1] *of reference in primary and secondary schools seem to be based on a parcelled out geometry GII (GII/GI) but pupils at both levels can succeed while working in a personal GWS being based on an assumed Geometry GI (GI/gII). It derives from this report that the primary school pre-service teachers work in a personal GWS near or identical to the GWS being used by the primary school pupil. Is this an ideal situation for pre-service teachers?*

INTRODUCTION

In the Elementary and Secondary School Levels curricula in the province of Québec, Mathematics is presented as one of the best subject for training in reasoning and argumentation, the example given being a geometry task. And doing so, the curriculum refers to a description of different sorts of reasoning in mathematics such as "*deductive, inductive or creative*" reasoning (p. 140).

In the introduction to the Secondary School Level curriculum, one can read that, thanks to mathematics, "*students continue developing the rigour, reasoning ability, intuition, creativity and critical thinking skills they began acquiring in elementary school*" (p. 183). More precisely, Geometry is a mathematical subject particularly relevant for hypothetical-deductive reasoning as:

> in Geometry, students use reasoning when they learn to recognize the characteristics of common figures apply their properties and perform operations on plane figures by means of geometric transformations. […]. They learn the definitions and properties of the figures they use to solve problems involving simple deductions.
>
> [...]
>
> - Observations or measurements based on a drawing do not prove that a conjecture is true, but must be used to formulate a conjecture.(p. 201)

But at both schools levels no example of geometrical work is given. So it is difficult to say what sort of deduction and sort of reasoning a pupil is supposed to have when facing a geometry task. Basically what kind of geometry pupils have to work in? And

[1] The expressions ETG of reference, suitable ETG and personal ETG, are to be taken within the meaning of Kuzniak (2009)

later on what kind of geometry are pre-service teachers mastering? To address these issues I will refer to the three geometrical paradigms as defined by Houdement-Kuzniak (2003) and the Geometrical Working Spaces as described by Kuzniak (2009).

To start with, I will give a very quick summary of the theoretical framework. Then I will analyze some tasks from Primary and Secondary textbooks along with the teacher's handbooks. This will lead to the issue I want to address: in which WGS are Primary School Pre-service teachers able to work?

THEORETICAL FRAMEWORK

Geometrical paradigms

According to Houdement-Kuzniak (2003), in the context of Euclidean Geometry which is taught in Primary and Secondary Schools, one can distinguish three paradigms. These paradigms can be characterized by their components: intuition, experience and deduction, the kind of space the pupil is working in, the status of the drawing and the privileged aspect of the drawing (or the object) for validation.

Table 1 gives the summary of the three paradigms as described by Houdement-Kuzniak (2003).

	Geometry I (Natural Geometry)	Geometry II (Natural Axiomatic Geometry)	Geometry III (Formalist Axiomatic Geometry)
Intuition	Sensible, linked to the perception, enriched by the experiment	Linked to the figures	Internal to mathematics
Experience	Linked to the measurable space	Linked to schemas of the reality	Logical
Deduction	Near to the Real, and linked to experiment	Demonstration based upon axioms	Demonstration based on a complete system of axioms
Kind of spaces	Intuitive and physical space	Physical and geometrical space	Abstract Euclidean Space
Status of the drawing	Object of the study and of validation	Support of reasoning and "figural concept"	Schema of a theoretical object, heuristic tool
Privileged aspect	Self-Evidence and construction	Properties and demonstration	Demonstration and links between the objects. Structure.

Table 1: Geometrical paradigms (Houdement-Kuzniak, 2003)

From different studies (Kuzniak, 2003; Kuzniak –Vivier, 2009; Kuzniak, 2010), it appears that in Primary School most curricula, textbooks and tasks in the classroom refer to GI. GII may appear from time to time in very specific situations. In Secondary School, it seems that geometry is living either in GI or in GII, the transition from one to the other being an important matter to address (Braconne-Michoux, 2008).

When considering a pupil or a mathematician working in Geometry, these paradigms were to lead Kuzniak to the idea of Geometrical Working Spaces. Moreover observations proved that the different paradigms were interlinked and the boundaries from one another were not clear cut.

Geometrical Working Space

In this paper the words Geometrical Working Space (GWS) will be understood in Kuzniak's meaning (2009):

> The Geometrical Working Space is the place organized to ensure the geometrical work. It makes networking the three following components: the real and local space as material support, the artefacts as drawing tools and computers (…) and a theoretical system of references possibly organized in a theoretical model depending on the geometrical paradigm.

Kuzniak (2009) refined the connections between the different paradigms according to their contributions to each other.

Assumed GI Geometry (GI/gII). In this geometry, the pupil is working on configurations from the real world, the validation relying on perception or measurements. But it may happen that theorems proved only in GII may be of use as "*technical tools avoiding measure or making calculations easier*".

Kuzniak presents an assumed GII Geometry referring clearly to Euclidean Geometry and its logical organisation. But in this geometry, theoretical properties emerge from intuition of space.

Parcelled out GII geometry (GII/GI). In this geometry, properties derive from GI experience. But here some hypothetical-deductive pockets can be developed using the properties already established.

Surreptitious GII Geometry (GI/GII). Adapting Kuzniak's definition of Surreptitious GIII Geometry, we may say that here, geometry teaching is drawn by more theoretical reasons aiming to GII, the pupil being left aside in the shift from one paradigm to the other. The textbooks may give us some examples of such geometry.

In the province of Quebec only approved textbooks are to be published and available in the classrooms. So we can assume that the interpretation of the curriculum and its expression in the textbooks are close. In other words, the referential WGS and the appropriate one are close to one another.

GWSs LIVING IN ELEMENTARY SCHOOL

If we are to quote the introduction to the curriculum, we may understand that during Primary School, a pupil works in GI then moves to GII:

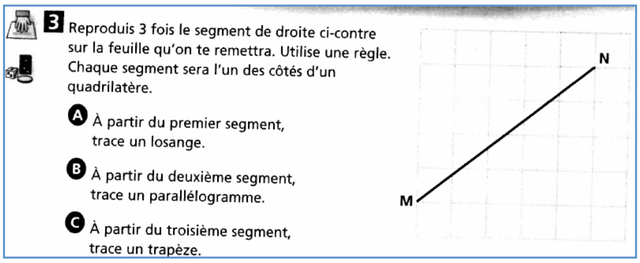
> Mathematics involves abstraction. Although it is always to the teacher's advantage to refer to real-world objects and situations, he/she must nevertheless set out to examine, in the abstract, relationships between the objects or between the elements of a given situation. For example, a triangular object becomes a geometric figure, and therefore a subject of interest to mathematicians, as soon as we begin to study the relationships between its sides, its vertices and its angles, for example. (p. 124)

If the teacher is to guide the pupils to abstraction, the meaning of "a geometric figure" has to be questioned. Is it as Laborde and Capponi (1994) meant it: a geometrical figure establishing the relationships between a theoretical object of geometry and the attached drawing? Or is it in a more general meaning where a precise drawing is made with geometrical instruments? This double-entendre is met further in the curriculum since the outcomes of year-2 are: "*the students* [...] *construct plane figures*..." (p. 147). And at the end of year-6, one can read: "[…] *the students* […] *can describe and classify plane figures,* […] *estimate, and measure or calculate lengths, surface areas …"* (p. 147). If we are to stick to the word "figure", we can suppose that there is a switch in the curriculum from the general meaning to Laborde and Capponi's meaning thus introducing a shift from GI to GII. But at the same time, the pupils can validate their answers mainly by measurement, and keep working in GI.

Furthermore according to tradition in teaching geometry in Quebec, students are working on instrumented drawings and the properties of the figures derive from their global aspect or measurement. So students mainly work in GI and gain some general knowledge from experience and "empirical generalizations". They called this information "properties" not knowing if they are theoretical or local. So if the WGS of reference at the end of Elementary School is probably a parcelled out GII Geometry (GII/GI) even a surreptitious GII one (GI/GII) insofar as proto-axiomatic considerations are to be taught whereas the supports on which the pupils work are drawings, are the pupils working in these very paradigms, the status of which being not clear and most of the validations being visual or instrumented as in GI.

To illustrate this interpretation of the WGS living in the classroom we will analyze some geometry tasks excerpt from two textbooks (year 4 and year 5) by the same authors.

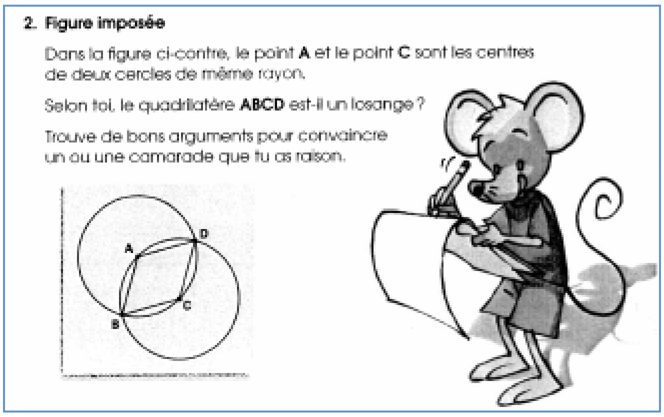
In figure 1 we give an example of a year-4 textbook (Clicmaths, 4[e] année). In task n°3, the questions can be answered in different geometries.



Figure[2] 1: Clicmath 4e année (p. 65)

Here the pupil works on a squared paper with instruments. He/she can validate his/her drawings perceptively thus working in GI. If he/she uses general properties when counting the number of squares in the grid or relying on axes of symmetry, he/she works in an assumed GI (GI/gII).

The example in Figure 2 comes from the year-5 textbook by the same authors.



Figure[3] 2: Clicmath 5e année (p. 53)

To answer this question, the pupil can say right away: "It's a rhombus. I can see it". But, because of the last sentence, he/she knows that such an answer is not allowed. So he/she can measure the lengths of the different sides and be convinced that it is rhombus. In both cases we can say that the pupil is working in GI. But as the last sentence suggests it, the pupil can guess that he/she must go deeper in his/her reasoning, reaching some part of GII, thus working in a parcelled out GI (GI/gII). The question which may arise at that time is: how the different reasons should be organized? Is there a first one and a second one or could they be worded in any specific order? According to the teacher's handbook, the expected answer is a formal

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[2] Free translation: "N°3: Copy this line 3 times on the (squared) sheet of paper you'll b given. Each line will be the side of a quadrilateral.

A. Starting from the first line, trace a rhombus.
B. Starting from the second line, draw a parallelogram.
C. Starting from the third line, trace a trapezoid.

[3] Free translation: "In the figure opposite, point **A** and point **C** are the centers of the two circles having the same radius. In your opinion, is the quadrilateral **ABCD** a rhombus? Find good reasons to convince a friend that you are right."

proof which is then clearly in an assumed GII (GII/gI): "*Points B and D are radiuses of the circle of centre A so AB = AD. ...*".

We can see there a shift from GI to some parcelled out GII Geometry and even an assumed GII but we cannot be sure that the pupils will be aware of this new paradigm. Are they able to make the distinction between the drawings they observe or they produce and the theoretical objects represented? We cannot be sure of that. This exemplifies how difficult it is to manage the shift from GI to GII (Braconne-Michoux, 2008). One could say that the different WGS living in an elementary classroom are: a WGS of reference which rests on a parcelled out GII geometry or a surreptitious one, a suitable WGS taking into account the WGS of reference but tending to GI and a personal WGS clearly in GI. But the shift from GI to GII is not clear: the drawing often has the status of a figure, without the pupil's knowledge.

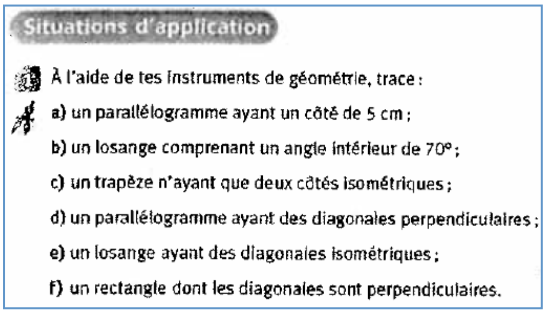
WGSS LIVING IN SECONDARY SCHOOL

The Secondary School Level curricula are in continuation of the previous ones.

> In geometry, the students make the transition from the observation to the reasoning. They state and use properties, definitions and relations to analyze and solve a situational problem. They construct figures if necessary, using a geometry set or dynamic geometry software. (p. 198)
>
> They learn the definitions and properties of the figures they use to solve of the problems involving simple deductions. (p. 201)

In this very curriculum, as a footnote related the contents of learning , one can read: "*In a geometric space of a given dimension (0, 1, 2 or 3), a geometric figure is a set of points being representing a geometric object such as a point, line, curve, polygon or polyhedron.* " (p. 216). Should we understand that a geometric figure is a theoretical object? Perhaps … The geometric properties the students must know at the end of each cycle are presented as "Principles of Euclidean Geometry" and are worded as geometrical facts: "*All the perpendicular bisectors of the cords of a circle meet in the centre of the circle*" (Cycle One, p. 219) or "*The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.*" (Cycle Two, p. 127). So it seems that the WGS of reference is based on GII. But what sort of GII: an assumed one or a parcelled out one? As we did before we will work on examples, excerpt from a textbook "Perspective Secondaire 1", by the same authors as "Clicmaths".

In activity n°1 (see figure 4), the student is asked to draw specific quadrilaterals.



Situations d'application

À l'aide de tes instruments de géométrie, trace :

a) un parallélogramme ayant un côté de 5 cm ;

b) un losange comprenant un angle intérieur de 70° ;

c) un trapèze n'ayant que deux côtés isométriques ;

d) un parallélogramme ayant des diagonales perpendiculaires ;

e) un losange ayant des diagonales isométriques ;

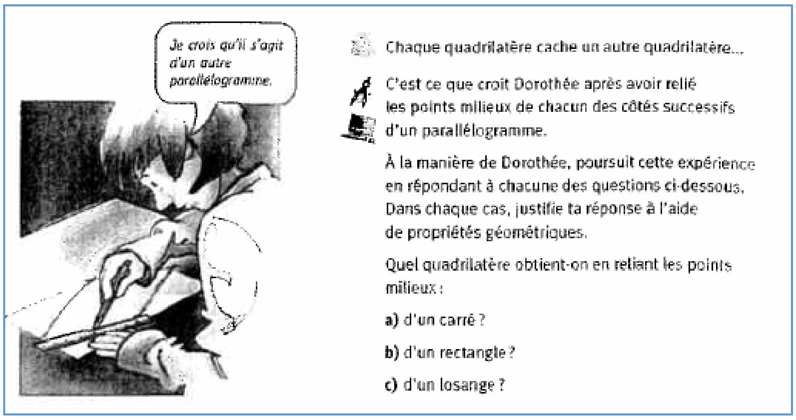
f) un rectangle dont les diagonales sont perpendiculaires.

Figure[4] 4: activité 1 p 89 (Perspective Secondaire 1 vol. A1)

From questions d) to f) the quadrilaterals have different names (ie: in f) a square is drawn). But these particular quadrilaterals are not questioned. Moreover, according to the teacher's handbook, the objective of this activity is to get more dexterity with the handling of the instruments. So this task has to be worked in GI. In our opinion, a good opportunity to move from GI to GII has been missed.

Further in the textbook, the student is asked to build different quadrilaterals using toothpicks and draw the figure he/she gets and justify the nature of the quadrilateral. In doing so the students go back to real objects and their validation are mainly perceptive. But as a justification is required, they have to refer and quote the properties of the diagonals of the quadrilaterals. Here we can consider that the students work GI using tools of GII, being in some parcelled out GII geometry.

Exercise n°9 (see figure 6), is very interesting since the property of the midpoints in a triangle is unknown at this school level, one can wonder in which paradigm the



Chaque quadrilatère cache un autre quadrilatère...

C'est ce que croit Dorothée après avoir relié les points milieux de chacun des côtés successifs d'un parallélogramme.

À la manière de Dorothée, poursuit cette expérience en répondant à chacune des questions ci-dessous. Dans chaque cas, justifie ta réponse à l'aide de propriétés géométriques.

Quel quadrilatère obtient-on en reliant les points milieux :

a) d'un carré ?

b) d'un rectangle ?

c) d'un losange ?

justification is expected, certainly not a formal proof.

Figure[5] 6: exercice n°9 p. 92 (Perspective Secondaire 1 vol. A1)

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[4] Free translation : "Using your instruments of geometry, trace:

a) A parallelogram with a side 5 cm long; b) A rhombus with an interior angle of 70°; c) A trapezoid having only two equal sides; d) A parallelogram having perpendicular diagonals; e) A rhombus having isometric diagonals; f) A rectangle whose diagonals are perpendicular."

[5] Free translation: "*Every quadrilateral hides another quadrilateral…*

That's what Dorothée believes after having connected the midpoints of every consecutive sides of a parallelogram.

The students can make a conjecture out of their detailed drawings. The best justification they can give is to report on the position of the diagonals of the new quadrilateral and derive its nature. So even if the question seems to be worded as in GII, the answer the students can give is an instrumented validation thus in GI. The expected justification given in the teacher's handbook relies on the lengths or the positions of the diagonals ("vertical" or "horizontal" as in their prototypic positions) is clearly worded as an assumed GI (GI/gII) task would be. Here is another example of the difficult transition from GI to GII.

In other textbooks students are asked to justify their conjectures using a list of theoretical properties (referring to GII). But they are not asked to organize their reasoning nor have to search theoretical reasons in order to build a convincing argumentation, along with a deductive reasoning. We can conclude that, though have been approved by the government, the textbooks offer very rare opportunities to work in GII and the suitable WGS rests at most on a parcelled out geometry GII, while pupils keep on working in a personal WGS resting on GI. Since, as far as geometry is concerned, the following years of Secondary School are dedicated to geometrical calculations but no formal proof, many students may leave Secondary School with no experiment of an assumed GII.

WGSs FOR PRESERVICE TEACHERS

Most pre-service teachers now University students have been trained according to these curricula. So we can assume that their personal WGS rests on an assumed GI geometry or at best a parcelled out GII one, very few of them being able to work in an assumed geometry GII. In other words, many students have a personal WGS which is not different from that of an elementary school pupil. We can fear that this WGS is likely not to be adapted to the teaching of geometry at Elementary School Level: pre-service teachers sharing their pupils' difficulties and their misconceptions.

In order to illustrate our words, we report two class experiments. On one hand, we asked students to do the activity illustrated in Figure 7, excerpt from a year-5 textbook.

Minh s'amuse à former des polygones à l'aide de cure-dents.
Comment doit-il disposer 4 cure-dents pour former un quadrilatère qui n'est pas un carré ?

Figure[6] 7: Presto 5^e année

As Dorothée did it, carry on with this experiment answering every question below. In each case, justify your answer using geometric properties. What quadrilateral do we get when connecting the midpoints
a) of a square? b) of a rectangle? c) of a rhombus?"

[6] Free translation :
Minh is enjoying forming polygons with toothpicks. How should he arrange 4 toothpicks to form a quadrilateral which is not a square?

To start with, several students found it difficult to switch from toothpicks to pens (no toothpick available). They kept on forming squares with the 4 pens in a prototypic position on their table. We had to help them to get to rhombuses. None of them tried to draw a schema of the situation. After discussion it appeared that this activity had highlighted the fact that several students were facing great difficulties in moving from real objects to represented ones and theoretical ones. These students' conceptions were still GI connected.

On the other hand, we asked them to draw the triangle as asked in figure 8, excerpt from Repères-IREM (1993).

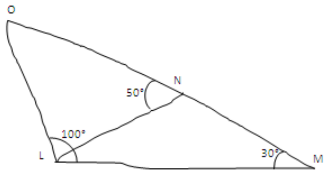
Problème :

OLM est un triangle. Le point N appartient au segment OM. De plus ∠ONL = 50° ; ∠OLM = 100° ; ∠OML = 30° et LM = 15 cm.

La figure ci-contre est mal construite ; elle ne correspond pas aux données.

Construire une figure en vraie grandeur.

Rapporter la démarche utilisée.



Figure[7] 8: construction of triangles

For many students, the process consisted in being very careful while using their instruments, keeping in mind that ∠ONL should measure 50°. Though all the students knew that the sum of the angles of a triangle is 180°, very few of them were able to use it in order to set up a reliable drawing process. Very few students noticed that the triangle OLN is isosceles in L and none put forward the idea that point N could be drawn using compasses. Most students had been working in an assumed GI (GI/gII), not an assumed GII (GII/gI), just as elementary pupils might do. Then one can wonder how as teachers they will look at their pupils' work and how they will be able to help them.

The origin of the situation is probably to be searched in the different WGSs living at Secondary School level. As we saw in the previous section they are not different enough from the WGSs living in Elementary School: tasks to be answered in an assumed GII (GII/gI) are very rare or non existent in some Secondary School textbooks. .

CONCLUSION

In conclusion, we can say that in either Elementary or Secondary Schools, the WGSs in use are based on an assumed GI geometry (GI/gII) tending to a parcelled out GII

[7] Free translation :
OLM is a triangle. Point N is on the line OM. Moreover ∠ONL = 50°, ∠OLM = 100°, ∠OML = 30° and LM = 15 cm. The figure opposite is badly drawn; it does not fit with the given. Draw a figure at real size. Tell the process you followed.

(GII/GI). The shift from GI to GII being difficult to deal with, more than often, the given tasks can be answered successfully by students working in GI. If a stress were to be on a real move to an assumed GII at Secondary School Level, we could hope that this would help our future pre-service teachers in geometry. How can we make students who work only in GI move to GII and make familiar with a new WGS close to the WGS of reference. The concern is that the contents of the handbooks are not always a relevant support and the duration of training is probably too short to let the students adapt to this new situation.

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