AMERICAN STUDENTS' DIFFICULTIES WITH THE CARTESIAN CONNECTION

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The purpose of the study was to gain insight into students' difficulties with linear functions, particularly epistemological obstacles with the Cartesian connection, the connection between the symbolic and graphic representation. Two cross-sectional studies were conducted with high school and college students. Participants completed several tasks and commented on their difficulties and successful algorithms. Interviews were conducted with some of the participants. Data analysis revealed that high school and college students performed poorly on tasks referring to the Cartesian connection. Moreover, findings suggest that even the few students who were successful at solving the tasks, did not have the necessary mathematical principles and coherence to organize and advance their structures of knowledge.

Keywords: Epistemological obstacles, Cartesian connection, linear functions

SIGNIFICANCE AND PURPOSE

Among the key algebraic concepts are linear functions (National Mathematics Advisory Panel, 2008). During the last two decades, student difficulties with linear functions have been studied by many researchers (Knuth, 2000; Lobato & Siebert, 2002; Moschkovich, Schoenfeld & Arcavi, 1993; Orton, 1983; Schoenfeld, Smith III & Arcavi, 1993; Stump, 2001; Zaslavsky, Sela, & Leron, 2002). Despite their efforts to gain insight into student difficulties and assist mathematics educators in implementing curricular and instructional changes, American students' difficulties with linear functions persist (American Diploma Project, 2010). Previous studies showed the importance of the Cartesian connection for student understanding of linear functions. The Cartesian connection states that a point (x_0, y_0) is on the graph of the line *l* if and only if its coordinates satisfy the equation of *l*, $y = mx + b$ (Schoenfeld et al., 1993), given that all the mathematical conventions associated with graphing in a Cartesian system of coordinates are respected.

The purpose of the studies reported here was to identify students' difficulties with the Cartesian connection, of interest being the "epistemological obstacles which occur because of the nature of the mathematical concepts themselves" (Cornu, 1991; Sierpinska, 1992), or those difficulties with knowledge that works satisfactory in old contexts but fails in new contexts (Brousseau, 1983).

Results from two cross-sectional studies are used.

OVERVIEW OF THE FIRST STUDY

The first study (Postelnicu, 2011) was conducted with 1561 Grades 8-10 American students enrolled in mathematics courses from Pre-Algebra to Algebra II [1], and their 26 mathematics teachers. All participants completed a Mini-Diagnostic Test (MDT) on aspects of linear functions, and commented on the nature of the difficulties. The MDT tasks, illustrating connections between various representations of linear functions, were those used by Greenes, Chang, and Ben-Chaim (2007) in their study with Algebra I students from United States, Korea and Israel. Across the three countries, the most difficult tasks were those requiring the identification of slope from the graph of a line. Students' difficulties with slope formula and the change in y and the change in x represented as line segments with oriented magnitudes is seen by Schoenfeld et al. (1993) as an example of missing the Cartesian connection. American students also had difficulties with tasks illustrating the Cartesian connection from point to line, asking students to determine whether a given point (x_0, y_0) is situated on a line, given the equation of the line, $y = mx + b$.

Semi-structured interviews (Goldin, 1999) with think-aloud protocol were conducted with 40 students. After completing the MDT, two students from each of the 20 teachers who agreed to be interviewed, were selected by the researcher and invited for interviews. In each pair of students, one student had the MDT total score above the group median, and one student had the MDT total score below the group median. Students were asked to present their solutions to the MDT tasks and comment on their mathematical difficulties. Student interviews were coded using open coding at the level of paragraph (Strauss & Corbin, 1998) and analyzed using the Linear Conceptual Field (LCF) framework, a theoretical framework inspired by Vergnaud's theory of conceptual fields (Vergnaud, 1994).

LCF is a set of situations that can be modeled using linearity and linear functions, schemes for dealing with situations, and sets of concepts and theorems necessary to analyze the operations of thinking, represented as a set of formulations and symbolizations (Vergnaud, 1994). LCF consists of:

i) Situations that can be modeled mathematically using linearity or linear functions. These situations reflect the current mathematics curriculum experienced by the American students, namely what Wu (2011) calls the Textbook School Mathematics.

ii) Schemes of actions, theorems in action needed to solve problems. Vergnaud's construct of scheme of action and the mechanism of knowledge development are borrowed from Piaget's genetic epistemology (Piaget, 1971).

iii) Representations of linear functions (e.g., tabular, graphic, symbolic) and mathematical formalizations.

We present here several examples of epistemological obstacles.

Task 1 asked students whether the point with the coordinates $(2,-8)$ is on the line $y = 3x - 14$. All students described their schemes of action without referring to the Cartesian connection as a formal mathematical theorem. More than 60% of the students (N=978) missed the Cartesian connection in Task 1. Students who missed the Cartesian connection from point to line failed to identify the ordered pair $(2,-8)$ as (x_0, y_0) in $y = 3x - 14$ or to check whether the statement $-8 = 3(2) - 14$ is true. They tried (unsuccessfully) schemes of action involving the construction of graphs or symbolic manipulations of the equation of the line. Particularly difficult was the construction of the line $y = 3x - 14$. Some students asked us to provide the graph of the line (e.g., "What graph?"). Other students could not graph the line $y = 3x - 14$ (e.g. "It's asking for a graph of the line $y = 3x - 14$. I know how to graph $(2,-8)$ but as the graph of the line…"). Even when successful at graphing the line and plotting the point, there were students who could not decide, based on their graphic representations, whether the point was on the line.

Task 2 asked to identify the slope of a line graphed in a homogeneous system of coordinates (see Figure 1).

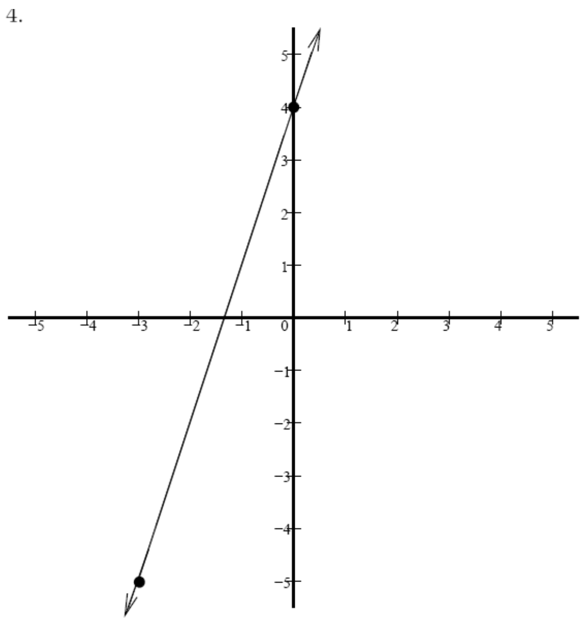

Figure 1: Task 2

To identify the slope in the geometric context from Task 2, one needs to know how to make a quantity. "Making a quantity" is the process of conceiving of a quantity as a quality of an object together with its magnitude. A magnitude is a numerical value assigned to a quantity by direct or indirect measurement. Measuring lengths on oriented axes of coordinates implies conceiving of an origin from where the measurement starts, an appropriate unit of measurement, and a sense of measurement, positive or negative (Freudenthal, 1983). In Task 2 one assigns the magnitude 9 for rise either by direct measurement (counting tick marks from $(-5,0)$ to $(4,0)$ on the Y-axis), or by evaluating the change in *y*, $y_2 - y_1 = 4 - (-5) = 9$. "Making a ratio" is the process of conceiving of a function of an ordered pair of magnitudes, while calculating a ratio is the numerical operation of calculating the value of the function for a particular pair of magnitudes (Freudenthal, 1983). One makes a ratio for slope by assigning $(9,3)$ to the ordered pair (rise, run), and calculates the value of slope by dividing 9/3. The main epistemological obstacle with the concept of slope in the situation from Task 2 was making a ratio. Some students could conceive of both rise and run, but failed to assign 9/3 to the ordered pair $(9,3)$ (e.g., "I can't remember if I'm supposed to add 9 plus 3 or multiply or divide …"). Other students conceived only of the rise, ignoring the run (e.g., "I made a line up to -3 and then counted over 3 from 0").

Task 3 (see Figure 2) is relevant because it brings in discussion the geometric perspective of slope (slope as a property of the line), and the analytical perspective of slope (slope as a property of the linear function) (Zaslavsky et al., 2002).

Task 3. The graph below represents the distance that a car travelled after different number of hours.

a) What is the speed of the car in part R?

b) What is the slope of part R of the graph?

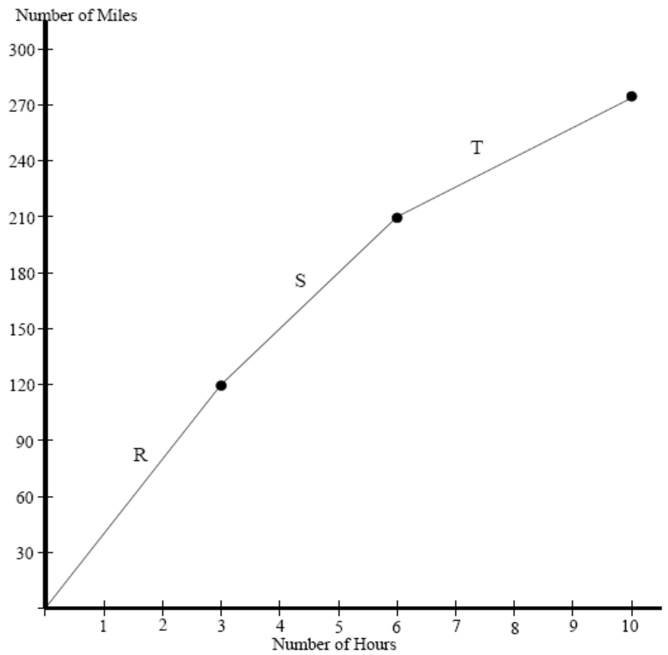

Figure 2: Task 3

Almost half of the students (N=696), correctly identified the speed of the car (*40 mph*). About half of the students who correctly identified the speed of the car, failed to identify the slope of the line segment (N=336). Moreover, about 82% of those students who failed to identify the slope of the line graphed in a non-homogeneous system of coordinates, correctly identified the slope of the line graphed in a homogeneous systems of coordinate on Task 2 (Postelnicu, 2011). One student expressed his difficulty this way: “In part *b* I didn't know whether to take from the values on the side or the units.” During the interviews, only the students who held a geometric perspective and not an analytical perspective of slope failed to take into consideration the values of the function for rise ($y_2 = 120, y_1 = 0$) when calculating $y_2 - y_1$, and instead counted the “tick marks” on the *Y*-axis. They calculated the slope of part *R* of the graph as “rise over run.” They found that the run, the line segment from the origin of the system of coordinates to (3, 0), had a length of 3 units on the *X*-axis, and the run, the line segment from the origin of the system of coordinates to (0, 120), had a length of 4 units on the *Y*-axis. They concluded that the slope of the line segment was 4/3 (rather than 120/3). When the analytic perspective of slope prevailed, and students used $\frac{y_2 - y_1}{x_2 - x_1}$ to calculate the slope, the interviewed students did not encounter difficulties.

OVERVIEW OF THE SECOND STUDY

The second study included 155 college students from a four year university, enrolled in five undergraduate mathematics courses (Intermediate Algebra, College Algebra, Precalculus, Brief Calculus, and Discrete Mathematics [2]). A task sequence on aspects of the Cartesian connection was administered to all participants. After completing the sequence, all participants were asked to compare the tasks (“Are the tasks alike or different? Explain.”) Semi-structured interviews (Goldin, 1999) were conducted with six students, randomly selected by the researcher. Three students had the total score above the group median, and three students had the total score below the group median. The interviewed students were asked to explain to an imaginary student not only how to solve the tasks, but also to justify why their algorithms work.

We discuss here two tasks:

Task 4. Find *m* and *b* such that the points $(1,4)$ and $(-2,10)$ are situated on the graph of the function $y = mx + b$.

Task 5. Determine *b* and *c* such that the points $(1,-3)$ and $(2,0)$ are situated on the graph of the function $y = x^2 + bx + c$.

Both tasks are applications of the Cartesian connection, but the mathematical object upon which the students had to operate was different, a linear function in Task 4, and a quadratic function in Task 5, respectively. In a Piagetian sense (Piaget, 1971), we wanted to see whether our students’ schemes of action were dependent on the mathematical object upon which they had to apply the Cartesian connection. In other words, our question was whether the students who applied the Cartesian connection in the case of the linear function from Task 4, could apply the same theorem in the case of the quadratic function from Task 5. The analysis of this epistemological obstacle had to be nuanced, because another question arose: Did the students who correctly solved Task 4 and/or Task 5 apply the Cartesian connection? To frame the discussion theoretically, Piaget (1971) discriminates between three types of knowledge: empirical knowledge abstracted from objects, pseudo-empirical knowledge abstracted from individual actions on objects, and reflective knowledge abstracted from coordinated actions on objects. The latter type of knowledge constitutes the basis for mathematical knowledge. The key difference between empirical and reflective abstractions is that the latter implies a projection from a lower to a higher level of knowledge, together with a reorganization of the entire knowledge in hierarchically superior structures. Given these theoretical considerations, we rephrased our question: What type of abstractions were employed by the students who correctly solved both Task 4 and Task 5? We considered evidence that reflective abstractions were employed if the student could generalize the Cartesian connection (e.g., a point (x_0, y_0) is on the graph of the curve *C* if and only if its coordinates satisfy the equation of *C*), apply it regardless of the appearance of the mathematical object *C* (e.g., symbolic representations like, $y = mx + b$ or $y = x^2 + bx + c$), and state it as the mathematical justification behind Tasks 4 and 5.

Only 31 students correctly solved Task 4. Eleven of the students who correctly solved Task 4, failed to solve Task 5. All the students who correctly solved Task 4, as well as the students who correctly solved both Tasks 4 and 5, compared the tasks by referring to the mathematical objects involved (e.g. linear functions, lines), and the actions performed (e.g., "You must use the slopes"). None of the students referred to the underlying mathematical justifications behind the task sequence, i.e., none of the students could justify their algorithms from a mathematical point of view. The interviews, coded (Strauss & Corbin, 1998) and analyzed using Piaget's theory of reflective abstractions, confirmed the procedural nature of students' knowledge (Skemp, 1976) and the lack of justificatory principles. Since none of the students could justify their successful algorithms, we concluded that no reflective abstractions were employed by college students.

DISCUSSION

Analyses revealed that students had difficulty with slope and the Cartesian connection. Notwithstanding that the content of the tasks referring to slope and the Cartesian connection varied across studies, the findings of the present studies and other studies (Orton, 1983; Lobato and Siebert, 2002; Schoenfeld et al. 1993; Stump, 2001; Zaslavsky et al., 2002) point toward epistemological obstacles.

Identifying the slope of a line graphed in a system of Cartesian coordinates seems to be a more difficult task than, for example, identifying the rate of change of a linear function when two instances of the varying quantities, (x_1, y_1) and (x_2, y_2), are given. One explanation may be that the graphing context for slope implies knowledge of mathematical conventions (e.g., if the oriented segment representing the rise is toward the positive sense of the *Y*-axis, its magnitude has a positive sign).Thus, the difficulty in Task 2 in identifying the slope of a graphed line lies with the connection with the graphic representation. Another explanation may be that the complexity of the problem increased for those students with a geometric perspective of slope. Indeed, calculating the slope of a graphed line as “rise over run,” implies knowledge of proportionality in the geometric context of similar triangles, or at minimum, coordination while identifying the rise and the run from a “slope triangle.” By contrast, students with an analytical perspective of slope, who calculated slope as “change in *y* over change in *x*,” or used the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$, had less difficulty since the connection with the graphic representation was reduced to identifying the coordinates of the points, (x_1, y_1) and (x_2, y_2).

Another possible explanation for student difficulties with slope may lie in the way this topic is presented in the U.S. mathematics curricula. For example, in one Algebra I textbook used by the first study’s participants, the slope is introduced in physical context, followed by the geometric context:

> Some roofs are steeper than others. In mathematics, a number called *slope* is a measure of the steepness of a line. The *slope of a line* is the ratio of *rise* to *run* for any two points on the line (Rubenstein et al., 1995, p. 361).

Usually, the geometric perspective of slope as “rise over run” precedes the functional perspective of slope as rate of change, the latter being introduced in advanced mathematics classes, like Precalculus and Calculus. Of note, the results of this study showed that only students with a geometric perspective of slope had difficulties determining the slope of a graphed line.

Insufficient presentation of the mathematical conventions behind the graphic representation of functions may be another explanation for some of the student difficulties. For example, in one of the Algebra I textbooks used by the participants in the first study, the graph of a function is not defined explicitly. Students are reminded in Chapter 1 about the procedure to graph a function, and an example is given, with no reference to the units of measurement for the axes (e.g., scale for the *Y*-axis):

> When you are given the rule for a function, you can prepare to graph the function by making a table showing numbers in the domain and their corresponding output values […] Let the horizontal axis represent the input *t* (in minutes). Label the axis from 0 to 5. Let the vertical axis represent the output *h* (in feet). Label the axis from 0 to 400. Plot the data points given in the table. Finally, connect the points […] (Larson et al., 2004, p. 49).

Difficulty with identifying the scale of the *Y*-axis in non-homogeneous system of coordinates was evident in Task 3. Students attempted to calculate the “rise” by counting the tick marks “by ones” for the *Y*-axis, instead of recognizing that each interval between two consecutive tick marks was 30 units.

Students had difficulty connecting between the symbolic and graphic representations in Tasks 1, 4 and 5. Knuth (2000) argued that the American curriculum and instructional approach emphasizes symbolic representations and manipulations, although symbolic representations encapsulate knowledge that students cannot successfully unpack (Kaput, Blanton, & Moreno, 2008). The graphic approach is often rejected by students and their teachers because graphic representations are not precise, and could introduce estimates and inaccuracies (Arcavi, 2003). The Cartesian connection necessary to solve Tasks 1, 4 and 5 can only add to the difficulty. But even when successful at solving Tasks 1, 4 and 5, students' knowledge was procedural, and mirrored their textbooks' knowledge - a collection of algorithmic steps for solving problems, lacking the mathematical foundations that justify the use of algorithms (Harel & Wilson, 2011; Postelnicu, 2011; Wu, 2011).

In short, the association between the epistemological obstacles and the geometric perspective of slope held by some of the students who encountered difficulties suggests that the precedence of the geometric perspective of slope, as well as the reconciliation between the two perspectives of slope, geometric and analytical, together with the mathematical conventions behind the graphic representation of functions, need to be addressed in our curriculum. The college students' difficulty to advance their knowledge by expanding their successful schemes, suggests that perhaps, even more important to be addressed is the need for active mathematical principles and coherence in our curriculum.

NOTES

1. In United States, linear functions are part of the mathematics curriculum for Grades 8-10. The usual high school course sequence is Algebra I (Grade 9), Geometry (Grade 10), Algebra II (Grade 11), followed by advanced math courses like Precalculus and Calculus. Of note, the current trend is to have Grade 8 students enrolled in Algebra I.

2. Upon college admittance, students may be tested and placed in remedial math courses like Intermediate Algebra, College Algebra or Precalculus.

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