IMPLICIT LEARNING IN THE TEACHING OF ALGEBRA: DESIGNING A TASK TO ADDRESS THE EQUIVALENCE OF EXPRESSIONS

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In this paper, I assume that some difficulties encountered by students, especially in elementary algebra, are generated by learning needs left implicit or unknown, inasmuch as they are not addressed within the learning institution. Taking an anthropological approach, I show that the equivalence of algebraic expressions, which plays an important part in managing, checking and anticipating algebraic transformations, is one type of learning left implicit in France. I design an experimental task to make students grasp this concept.

Keywords: implicit learning, equivalence of algebraic expressions, school algebra

ORIGIN OF THE STUDY

My questions about the existence of implicit learning needs left out of the teaching of algebra in France has its origins in a larger line of questioning addressed in my PhD: what learning materials should be given to teachers to help them identify and manage difficulties encountered by students in school algebra, and develop the students' personal relationships with algebra? (Pilet 2012) I addressed this question within the framework of the anthropological theory of didactics (ATD) (Chevallard, 1999) in order to analyse and understand the conditions of production and circulation of learning in institutions. I believe that *some difficulties encountered by students* in algebra may be intensified by the fact that students learn within institutions where certain conditions and constraints are applied to the transmission of knowledge (Bosch, Fonseca, Gascon, 2004; Castela, 2008). Specifically, I made the following assumptions. On the one hand, some students' difficulties are generated by learning needs that are unknown to teachers or are left implicit, inasmuch as they are not addressed within learning institutions. On the other hand, making resources available to teachers and students to organize implicit learning can promote the evolution of students' personal relationships with algebra.

In this article, I present one of these implicit learning needs: the equivalence of expressions and a task allowing it to exist. The institution in question is secondary school, toward the end of compulsory education in France (14-15 year-olds). First of all I show that equivalence of algebraic expressions is both implicit in teaching algebra in secondary education and necessary in order to conduct and check algebraic transformations. Then I propose a particular learning situation asking the following question: what types of tasks should we design in order to allow the equivalence of algebraic expressions to exist?

THE NOTION OF PRAXEOLOGY AS A FRAMEWORK

The ATD (Chevallard, 1999) uses the term "praxeology" to refer to any human practice and, in particular, to mathematical activities.

> "The term of 'praxeology' […] enables us to consider two terms that are often opposed within the same entity: the 'practical block' or know-how and the 'theoretical block' or knowledge made of the discursive elements used to describe and justify the practice. A praxeology is made of four components: type of task, techniques, technologies and theories. The *praxis* or 'practical block' contains a set of *types of tasks* to be carried out and a set of *techniques* to do so, 'technique' being considered here in a very general sense of 'ways of doing'. The *logos* or 'theoretical block' is made of a double-levelled discourse. A *technology* or 'discourse on the technique' to explain what is done, to left other interpret it assumptions that validate the technological discourse and organise the praxeological elements as a whole, form what we call theory." (Bosch, 2012)

I use the notion of praxeology as a framework. On the one hand, to identify learning needs often ignored by the institution and often left implicit in the curriculum and textbooks (Bosch, Fonseca, Gascon, 2004; Castela, 2008), such as the equivalence of expressions, I have linked epistemological reference praxeologies and praxeologies of teaching (curriculum, textbooks). On the other hand, I show that equivalence of algebraic expressions is an element of the theoretical block allowing the justification and checking of techniques of algebraic transformations classified as different types of tasks: "develop", "simplify", "factorize a given type of algebraic expression".

EQUIVALENCE OF ALGEBRAIC EXPRESSIONS: EPISTEMOLOGICAL ASPECTS

The equivalence of expressions is defined as follows: two algebraic expressions are equivalent if and only if they produce the same number for any letter value.

The notion of equivalence of expressions is present in several studies, where it appears as an indispensable theoretical element of the conduct and checking of algebraic calculations.

Frege: *sense* and *reference*

Equivalence of algebraic expressions refers to the distinction established by Frege (1971) between *Sinn* and *Bedeutung*, usually translated, respectively, as *sense* and *reference*. Drouhard (1992) makes the same distinction for mathematical objects. A mathematical object has a unique denotation but may have different senses. For example, the expression $4(x-1)$ can be written as $x^2-(x-2)^2$ or as $4x-4$. These expressions have different *senses* but have the same *reference* as they refer to the same number. Algebraic transformation is a subtle process moving between *sense* and *reference*. The choice of transformations of expressions is checked by *sense* (the final goal) but respect for written denotation is an indispensible criteria. When one expression is transformed into another, the goal is to obtain two equivalent expressions (that are equal for all values), that is, which have different *senses* but the same *reference*.

For many students, however, the conservation of *reference* during the transformation of an expression is unknown or absent. For instance, when a teacher addresses a common error $(x+4)^2=x^2+16$ and uses a numerical counter-example (for $x=1$, $(1+4)^2=25$ is different from $1^2+16=17$), students may not convinced because the argument used (implicitly) refers to the conservation of *reference*. Similarly, when a teacher suggests the identity $(a+b)^2=a^2+2ab+b^2$ to replace the student's $(a+b)^2=a^2+b^2$, there can be misunderstanding. The teacher is making reference to *reference*, while for students it may seem to be a "choice" made by the teacher.

Kieran: theoretical control and the equivalence of expressions

From an international synthesis of research related to the learning of algebra, Kieran (2007) puts the equivalence of expressions at the core of the theoretical elements of transformational activity. This activity involves the use of transformational rules (factorization, expansion of products, rules for solving equations and inequalities, etc.). It is one of three complementary activities distinguished in the GTG model of conceptualizing algebraic activities developed by Kieran: generative activity, transformational activity and global/meta-level activity.

> "One resource of algebra is a rich plurality of symbolic forms; one core notion, that of equivalence. Equivalence and transformation are linked notions, indicating sameness perceived in difference for some purposes, or indifference with respect to others? The existence of multiple expressions "for the same thing" can suggest the very possibility of transforming expressions directly to get from one to another." (Kieran 2007, p.722)

According to Kieran, equivalence of expressions has a fundamental role in theoretical control, ensuring that the transformed expression is equivalent to the second. This verification can theoretically be done in two ways: either by reference to the properties of algebra used or by linking with numerical and substituting numerical values for letters. However, students have a great deal of difficulty in identifying the properties they use when they transform algebraic expressions (Kieran, 2007) and, in France, students have difficulty making the link between transformation of expressions and substitution of numerical values. Students have difficulty making connections between the arithmetical and the algebraic world.

The equivalence of expressions appears in these studies as an element of the theoretical block of praxeologies related to the transformation of expressions in order to direct, anticipate and verify transformations of algebraic expressions. Later on, it also leads students to the conclusion that one of the expressions can be chosen (work on the meaning of expressions) according to one's ultimate goal (choosing the expression that can be best used to solve an equation, calculate the antecedents of a function, calculate the image of a function).

EQUIVALENCE OF ALGEBRAIC EXPRESSIONS: AN IMPLICIT LEARNING IN THE FRENCH CURRICULUM

I analysed the curriculum and several secondary school textbooks in France (for 14-15 year-olds) in order to identify the presence of equivalent expressions as a technological-theoretical element of the transformation of expressions. The French secondary school curriculum addresses in a formal setting how to develop, factorise and simplify expressions as a goal in itself. The ideas of getting students to perceive the fact that two different expressions may represent the same object, and the idea of checking calculations, are not included in the curriculum and very rarely appear in textbooks. It is for this reason that I have designed a task around the goal of studying equivalent expressions.

DESIGN OF A TASK

A task about conjecture and proof

The task focuses on the fundamental question: What are equivalent expressions? As the curriculum doesn't deal with the notion of equivalence, I ask students the following question: "Are these expressions equal for any value of x?". The type of task is T: "Prove that two expressions are equal for any value of the letter".

The following exercise is selected:

Exercise C group *(respectively B group)*

We would like to know if the following three expressions are equal for all x:

- $A(x)=(x+2)^2-4$ *(respectively $(x-1)^2-4$)*
- $B(x)=x(x+4)$ *(respectively $(x+1)(x-3)$)*
- $C(x)=9x-6$ *(respectively $x(x-2)-x^2-2$)*

x	A(x)	B(x)	C(x)
2 *(respectively x=1)*			
3 *(respectively x=-1)*			
0 *(respectively x=0)*			

1) Calculate the value of three expressions for x=2 and x=3 *(respectively x=1 and x=-1)*. What can you guess about the equality of these expressions?
2) Calculate the value of three expressions for x=0 *(respectively x=0)*. Does this confirm your conjecture? Otherwise makes a new guess.
3) Are the three expressions equal for any x? Justify. Does your guess hold up?

The exercise consists of three questions. Three expressions are given. Expressions A and B are equivalent but not C. The first two questions ask students to make a conjecture about the equality of the expressions for any value. This step is absolutely necessary in order to bring students to understand that two expressions can produce the same value. First, students test expressions for two numerical values chosen so that they return the same number. This leads them to formulate an initial hypothesis: the expressions are equal for any value of the letter. Then, a third test eliminates expression C and formulates a conjecture on the equality of any value of expressions A and B. The final question asks students to prove this conjecture. It involves algebraic proof and the use of a numerical counterexample.

This task is unusual in the current curriculum. First of all, the link with numbers in order to test the expressions is seldom used. In addition, students are responsible for recognizing which proprieties to use and for deciding to transform expressions A and B so as to prove their equivalence. The proof task forces students to finalize the algebraic transformations and to justify them, which they do not do in the technical exercises.

A differentiated task

The task I have presented comes from my Ph.D. thesis, which addresses differentiation of teaching. For this reason, I present two tasks which are similar but have different algebraic expressions for two groups of students, referred to as groups B and C. Teaching is differentiated in the following way: the learning objective is the same for the whole classroom—all students work on the same type of tasks—but the task is adapted to students' learning needs as identified by the diagnostic assessment *Pépite*[1]. Since differentiation of teaching is not of this article's main subjects, I will not go further into our choices here. For more information, please see (Delozanne and al., 2008; Pilet, 2012; Grugeon and al. 2012).

A priori analysis

Praxeology analysis

The type of task is T: "Prove that two expressions are equal for any value of the letter". It is possible to use several techniques involving different technologies. For example, students can transform the expressions and compare the coefficients of the same degree. In this case, the theoretical block references the properties of polynomials: "two polynomials are equal if and only if they are of the same degree and their coefficients of the same degree are equal." The technique I want to use consists of conjecturing equivalence using numerical tests and then proving it through a counterexample or an algebraic proof. This technique can lead students to grasp the *reference* of algebraic expressions at a technological level. In addition, it can be reapplied in later lessons in order to verify algebraic transformations.

The role of numerical tests and the table of values in the conjecture

The use of numerical tests aims to highlight the fact that the expressions either do or do not produce the same number. The layout of these tests in a table of values is a vital aspect of the task. It emphasizes the fact that two expressions can have the same *reference*.

Initially, I planned to use a spreadsheet (or a table of values) in collective phases after question 2 in order to emphasize the fact that expressions A and B refer to the same number. This option was abandoned because the teacher participating in the experiment did not have the necessary equipment. In our conclusion, I will speak about the implications of this choice on students' approaches to the task.

Moreover, students are allowed to use calculators, because it is important to avoid difficulties in numerical calculations (which is not the target of the lesson).

The role of quantification in showing students the necessity of proof

After noticing that expressions A and B are equal for several values, some students may suggest an incorrect technique, such as “the expressions are equal for one or two values so they all are equal". The teacher is expected to rely on quantification in order to counter them: “we want a proof for any value”. The use of quantification can help students understand the necessity of proof and use algebraic calculations. The teacher may highlight that it is impossible to numerically test every possible value. Although requiring formalism is obviously not one of the task’s goals, it remains a key component of students’ being able to master the notion of equivalence of expressions.

An algebraic proof requiring students to choose the properties to be applied

Students are expected to use a counter-example to prove that expression C is not equal to A or B. To show the equivalence of expressions A and B, development is expected. In order to do this, students must detect the structure of the expressions and choose the correct rule to apply.

Predicted teaching scenario

The development of the activity involves individual work phases and collective debate phases. The fact that groups are working on different expressions is an opportunity to contextualise the task. At the end, the institutionalization focuses on *reference* of algebraic expressions and links it with a technique to check calculations by numerical substitutions.

EXPERIMENTS

This experimental task was tested in a *troisième* class (ninth grade, 14 year-olds).

Context

I worked throughout the 2011-2012 school year in collaboration with several secondary school teachers to test the viability of our proposals in ordinary classrooms and allow them to be coherently integrated with teachers’ educational projects. The experiment that I present here was conducted by one of these teachers. This collaboration allowed us to design the scenario and discuss all of the tasks’ consequences together.

The experiment’s protocol was specific to this research project. The teacher first had her students take the *Pépite* diagnostic assessment in order to better understand their difficulties and to design a differentiated lesson. Her class was divided into two groups: the C group (15 students) and the B group (6 students). She covered content related to algebra by introducing a new property of the calculation ($(a+/-b)^2=a^2+/-2ab+b^2$ and $(a+b)(a-b)=a^2-b^2$) and solutions to several generalisation and proof problems. The given task is located at the end of a chapter on algebraic expressions. Students had already frequently encountered type T tasks through questions asking “are the calculation programme equivalent?” or “are these equality true or false for any value?” (for example $a^2=2a$, $3+4a=7a$).

Data analysis methodology

I filmed, recorded and translated[1] individual and collective discussions between the teacher and her students. I collected and then analysed the students' papers, but I will not present these analyses in this article.

The data analysis is based on the *a priori* analysis of the task. Across the various phases of debate, individual work and institutionalisation, I check whether the table of values, the conjecture and quantification allow students to understand that two expressions can be equal for any value.

A posteriori analysis

The task was given during a 50-minute class period. To begin the exercise, the teacher quickly told students the lesson's goal: "We'll be working on equal expressions." She emphasized substitution with numerical values, but did not reveal the goal of this substitution. This may explain why at the beginning students focused on the simple and isolated task of substitution by a numerical value. Errors appeared like concatenation (3+2a=5a), ignoring rules for parentheses and the order of operation. But the teacher enriched the lesson by using calculators, which spared students some of the work of doing numerical calculations and focused the lesson on the task's ultimate goals. Thus, the choice of appropriate numerical values, the table of values and the use of calculators favored the evolution of students' activity toward work on the *reference* of expressions. After focusing on the simple, isolated task of substitution, the students were surprised by the fact that different expressions return the same value.

Chloé (C): But **in fact, it should give the same results everywhere**?

Teacher: Well, sometimes it… ? [to Yann] Ok. So, what did you get, here? 12, 12, 12. And here, 21, 21, 21. So, in theory, what should we say, then?

Yann (C): That they… they… they are all equal.

Teacher: Yeah. The expressions are equal.

Yann: But I don't know.

[…]

Teacher: Yes, Mélusine?

Philinte (B): We got -2.

Mélusine (B): **Nobody got -2 for the third expression. Nobody got the same thing**.

Through these discussions, students begin to address the *reference* of algebraic expressions which allowed the conjecture about the equivalence of expressions to exist and evolve. The challenge that follows the conjecture is to show students the need for algebraic proof. I show that the use of quantification has played a key role.

Teacher: No, we don't know, do we? We just showed that it was true for two values. It doesn't necessarily mean **that it was true for all values**. So, now, you calculate for x equal to zero. […]

Teacher : Here we are. So, let's go, explain that! And so, your third expression, what is it? You will have to prove **that all of those expressions are equal for any value.** So, what are you going to use to be able to prove that these expressions are…?

Teacher: A and B. So. How shall we manage **to prove that it's true for any value**?

In addition, quantification has played an important role in an attempt to show students the limits of their own technique: "it is true for two values so it must be true for all values".

Teacher: How did you manage, Ina, to prove…? Does the table allows us to assert that the expressions are equal?

Students (C): No.

Ina (C): Well, yes, because…

Théodule (C): No. Because we didn't use the distributive property.

Teacher: I do agree with you. Here, we find out that these are the three same values. But if you find out that both expressions are equal for three values can you assert that **they are equal for any value**?

Students (C): No.

Teacher: No. So, how can you manage to prove that your expressions **are equal for any value**?

Ina (C): Well… we calculate again.

Teacher: We should calculate again for another value?

Ina (C): Well, yes. Well…

Teacher : If you calculate again for another value, **it will mean that they are equal for four values. You, you want to prove that they are equal for any value**.

The teacher's role allowed most students to understand the necessity of the algebraic proof. Fourteen students out of eighteen suggested an algebraic proof in order to show that A is equivalent to B but only nine students used a counterexample in order to prove that C is not equal to A and B. However, I observe that students have a quite difficult time recognizing the structure of the expressions and the properties to be applied.

DISCUSSION AND PERSPECTIVES

To conclude, I return to the goal of the task. Did it successfully lead students to understand the *reference* of algebraic expressions, and thus recognise equivalent expressions? Since our analysis is a case study, the conclusions to be drawn from it should be limited.

The choice of appropriate numerical values—the table of values—allowed students to understand the fact that expressions can produce the same number. I believe, however, that presenting a table of values with infinity of numbers when the teacher asks students how they approached the problem could have reinforced students' understanding of the situation. The use of a spreadsheet in collective phases could play an important role in emphasizing the *reference* but differs too much from teachers' current practices. The analysis of discussions shows that the *reference* and the equivalence of expressions could be due to numerical conjecture, but the teacher did not suggest it again in question 3 or during the institutionalisation. The institutionalisation dealt with part of this content. The technique of numerical conjecture and algebraic proof, aiming to prove that two algebraic expressions are equal (or unequal) for any value of the letter, was presented. But the teacher did not add anything that might help students draw conclusions about the *reference* of expressions. The fact that algebraic expressions with two different written forms give the same number for any value was not presented.

This experiment shows the real potential of the designed task, as long as work has been done beforehand with teachers emphasizing their role and the task's goals. Readers should keep in mind that the designed task involves very few aspects of the current French curriculum (equivalence, links with numerical or quantification) and is quite different from teachers' current practice. The implicit learning that is essential to educational algebra needs a long period of preparation with teachers on two points: on the one hand they must be aware of the links between implicit or ignored learning and students' difficulties in algebra and, on the other hand, they must develop their algebraic teaching practices. For this reason, I continue to work collaboratively with teachers to accompany the task with a discussion of the didactic issues at hand and their management in the classroom.

NOTES

[1] At present, the *Pépite* software is developed in PepiMeP project which consists of implementing computer resources in *LaboMep* platform, which has been developed by *Sésamath* to help teachers to differentiate students' learning in elementary algebra. Ile-de-France Region supports the project. *Sésamath* (http://www.sesamath.net/) is a French mathematics teachers' association, which has a central place in French online database systems. More information is given in a paper in working group 15 "Bridging diagnosis and learning of elementary algebra using technologies".

[2] In this paper, I translated transcripts from French to English.

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