GENERALISING THROUGH QUASI-VARIABLE THINKING: A STUDY WITH GRADE 4 STUDENTS[1]

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This communication presents a study from the beginning of a teaching experiment to promote grade 4 students' algebraic thinking. It aims to analyse how quasi-variable thinking contributed to the development of generalisation and to the first uses of symbolisation by the students. The data were collected from two mathematical tasks that explored computation strategies. The lessons were taught by the researcher (the first author), the data were collected by video recordings and the collective discussion moments in the classroom were analysed. The results show how students use particular numerical expressions to generalise the relationships underlying the structure of the calculation strategies. Thereby, using quasi-variable thinking, students express the generalisation in natural language and start making a pathway to symbolisation.

Key-words: algebraic thinking, generalisation, symbolisation, quasi-variable thinking

INTRODUCTION

The development of algebraic thinking from the first years of schooling should be understood as a way of thinking that brings meaning, depth and coherence to the learning of other topics and has the potential of unifying the existing mathematics curriculum (National Council of Teachers of Mathematics, 2000). The new Portuguese curriculum (Ministério da Educação, 2007) assumes that students should start to develop the algebraic thinking by using arithmetic as an entry point, as they work with generalisable regularities in numbers and operations and also by the study of figurative sequences.

The present communication aims to discuss how the use of quasi-variable thinking can contribute to the development of generalisation and to the beginning of symbolisation. Namely, we seek to understand: (i) How do students come to generalise computation strategies from particular numerical expressions?; and ii) How is generalisation starting to be expressed by students into symbolic mathematical language?

THEORETICAL BACKGROUND

According to Blanton and Kaput (2005), algebraic thinking can be regarded as "a process in which students generalise mathematical ideas from a set of particular instances, establish those generalisations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways" (p. 413). Considering the potentially algebraic character of arithmetic as one of the possible approaches for the development of algebraic thinking, the construction of the generalisation can be promoted through the exploration of numerical relationships and arithmetical operations and their properties and, also, by developing the notion of equivalence related to the equal sign (Carpenter, Franke & Levi, 2003).

Rivera (2006) suggests that numerical systems should be taught in a way that students understand the numerical relationships and proprieties of individual objects and progressively realise that those are invariant independently from the considered objects. The regularities that students find on arithmetical operations can be the basis for the exploration of generalisation about numbers and operations and also to practices as the formulation, test and proof of the produced generalisations. In this way, particular numerical expressions can be used to work general relationships. Fujii (2003) uses the expression of *quasi-variable* to describe a "number sentence or group of number sentences that indicate an underlying mathematical relationship which remains true whatever the numbers used are" (p. 59). Within this perspective, students can use generalisable numerical expressions, focusing their attention in the expressions' structure, and identifying and discussing the algebraic generalisation before being introduced to formal algebraic symbology. This kind of quasi-variable thinking (Fujii & Stephens, 2008) can provide an important bridge between arithmetic and algebraic thinking and, also, a gateway to the concept of variable (Fujii, 2003).

Britt and Irwin (2011) consider that algebraic thinking should provide opportunities for all students to work with several layers of awareness of generalisation. These authors suggest that a pathway for algebraic thinking develop in such a way that "students use three semiotic systems to express that generalisation: first they should work with numbers as quasi-variables, then with words and finally with the literal symbols of algebra" (p. 154). Similarly, Russell, Schiffer and Bastable (2011) advocate the introduction of algebraic notation when students already express their ideas into words and images allowing them to access the meaning of symbols. These authors contend that this new form of representation, not only provides a concise expression of students' ideas, but also offers new ways of perceiving mathematical relationships.

In this communication, we also assume the conception of generalisation as a dynamic and social situated process that can evolve through collaborative acts (Ellis, 2011). In this perspective, the classroom situations are seen as multiple process of interaction "in which the students and the teacher co-contribute to the development of meaning through their talk, shared activity, and engagement with artefacts" (Ellis, 2011, p. 311). This interactionist perspective includes both teacher-student interaction and student-student interaction and allows researchers to take into account how shared ways of interacting promote the development of generalisation.

Cobb, Bough, McClain and Whitenack (1997) designate *reflective discourse* as a kind of a classroom discourse in which mathematical activity is objectified and becomes an explicit topic of conversation. When students are engaged in a collective act of reflective discourse, they have the required circumstances for mathematical learning, and their individual contributions develop the discourse that supports and sustains collective reflection. These authors recognise that students' mathematical development "is profoundly influenced both by the face-to-face interactions and by the cultural practices in which they participate" (p. 271).

METHODOLOGY

The results presented in this communication are part of a broader study which focuses on the implementation of a year-long teaching experiment (Gravemeijer & Cobb, 2006) which aims to promote the development of algebraic thinking of grade 4 students. The teaching experiment took place in the school year of 2010/11 and the mathematical tasks proposed to the class drew on the mathematical topics defined by the annual plan made by the school teacher. However, these tasks were innovative considering the usual teacher's practice as they accommodated the prospect of conceiving the algebraic thinking as guiding the syllabus (NCTM, 2000), through a logic of curricular integration. Taking as a planning starting point some insufficiencies detected on students' number sense, we developed a sequence of tasks, focused on the exploration of numerical relations and operations properties, and had as goals the identification of regularities and the expression of the generalisation through natural language, and the beginning of a way towards mathematical symbolisation. The use of some informal symbolism was introduced, particularly, in the tenth task, when the teacher-researcher proposed the use of the symbol "?" to express "what is the number that..." in expressions like this "?x5=100".

In this communication, we focus on the moments of collective discussions in the classroom, after the students' work in pairs on two of the mathematical tasks in the teaching experiment (the 12th and the 14th). These tasks explored computation strategies with the goal to lead students to express the generalisation in natural language and to begin also to express it in mathematical language. These were the first tasks of the teaching experiment where the teacher-researcher intentionally promoted the expression of generalisation into mathematical symbolic language. The lessons were videotaped, and from the analyses of the videos we choose the episodes that show how the exploration of particular numerical expressions is fostering the generalisation of the computation strategies involved in each of the tasks and also the way students are starting to use symbolic mathematical language corresponding to the teacher-researcher's challenge.

RESULTS

The first task analysed in this communication – *Calculation using double* – explores the double and half relationships between the 4 times table and the 8 times table (fig. 1). Its intention was that students explain the relationships and generalising the strategy, in natural language and the translation to mathematical language.

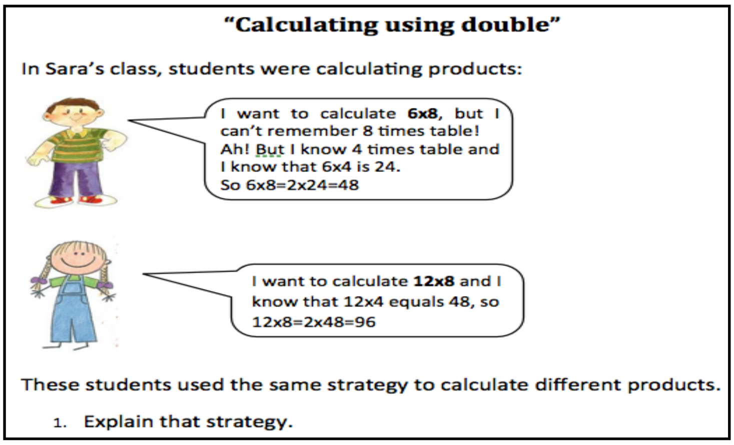


Figure 1 – The task "Calculating using the double".

In the collective exploration of the task, after the discussion about the correctness of the expression, some particular cases of the computation strategy by the students are presented. Next, the teacher conducts the discussion with the goal of making the students identify the strategy that is used by extending the strategy beyond the studied cases in order to induce the students to express it in a generalised way.

Teacher: Okay. We have three examples, but does this strategy only fit those examples?

Students: No.

Fábio: The strategy we used is good for all computations.

Teacher: And how can we synthesise that strategy in a clear way? What strategy was that?

Diogo: We made the double computation.

Teacher: The double of what?

Diogo: Double of the result.

Teacher: Can you explain better? Develop it a little more…?

(Diogo does not answer)

Teacher: What was the multiplication table that we want to work on?

Students: The 8.

Teacher: And to work the 8 times table, we used which multiplication table?

Students: The 4's.

Rita: We can use the halves.

Teacher: And what did we find out? I can make 8 times table using which one?

Students: 4's.

In this episode, these students reveal that they understand that there is a property involved in the numerical equalities present in the task. But in this moment they still refer it as a procedure they can apply to all numbers.

After that, a student, Rita, was able to express the computation strategy of generalisation beyond the particular cases, but still using a confusing and repetitive language. Having this in mind, the teacher asks if the expression should be simpler and clear, forwarding the students in this process.

Rita: If we go to 4 times table, the multiply 4 by the number that we wanted from 8 times table, if we multiply twice, we will have the result from 8 times table.

Teacher: How can I say that in a simpler way?

Carolina: To know 25x8 we do from 25x4.

Teacher: You are using a particular example. But what if it's more general? We were talking about the 8 times table and the 4 times table. I can say that in a very simple way. To know the 8 times table, what do I do?

Students: Double the 4 times table.

From this moment on, one student proposes the generalisation of the computation strategy in natural language and writes it on the board: "To find the 8 times table, we do the double (x2) of the 4 times table". Several students show they identify the underlying relation between these two tables.

After that, the collective discussion was conducted to enable students to express the generalisation in mathematical language. As this was the first time that students were confronted with this issue, the teacher attempts to make them understand what it means to write the generalisation in "mathematical language".

Teacher: Now I want that you think about the sentence João wrote on the board and try to write it in mathematical language. How can we use mathematical language?

Students: With operations.

Teacher: So, how can I write that? But pay attention because I don't want particular cases like 6x8, 12x8 or 25x8, I want that to all numbers of the 8 times table and 4 times table.

Rita: We can do it to 7x8.

Teacher: But that is a particular case. I want to all cases.

Rita: How is that?

Teacher: To all cases in 8 times table. What happens in 8 times table?

Fábio: Is always plus 8.

Teacher: Ok, it's always plus 8. But if we use multiplication, what are we doing?

Students:	We are always multiplying by 8.
Teacher:	How can I write that?
Rita:	We can use a question mark.
Fábio:	Times 8.

The expression of generalisation in mathematical language was not immediate. First, the students tried to use particular examples. Then, the teacher conducted the class to use the structure of the computation strategy of the multiplication by eight to all cases. At that moment, when Rita said "question mark", she uses the symbol that she knew from the tenth task, briefly described before. Then, Rita went to the board and wrote the expression shown hereafter:

Figure 2 – Rita's generalised expression in mathematical language.

Then, instead of saying something about the correctness of the expression, the teacher suggests the substitution of the symbol by a specific number. In this way, the teacher intends to support the students in attaching meaning to that symbol and in finding out if the expression written by Rita was correct. Some students suggest the use of number six and then others numbers. In this way, the students realised that one expression like 6x8=2x6x4=6 is not correct, and concluded that following the last equal sign there should not be the question mark. As a result of that, most of the students expressed that they agree that the right expression should be the following one:

Figure 3 - Final expression of the generalisation in mathematical language, made collectively.

The second task analyzed in this communication – *Afonso's strategy* – concerns the inverse computation strategy of the previous one: multiplying by five is equivalent to the half of the multiplication by ten (fig. 4). The particular case of 36x5 is proposed in the task as it follows:

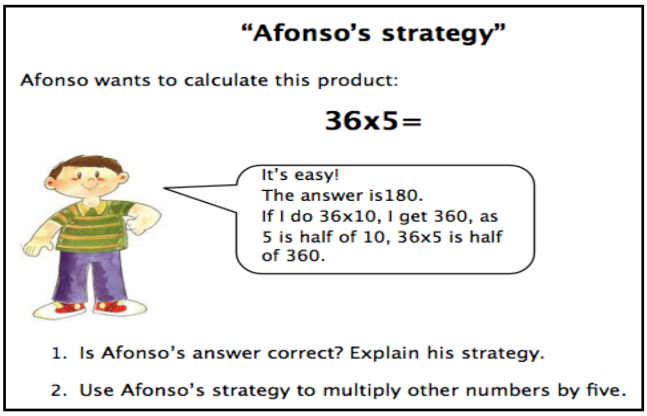


Figure 4 – The task "Afonso's strategy".

In the beginning of the collective discussion of the task, the teacher asked the students to express the Afonso's strategy in natural language. Students did it without difficulty, expressing it the following way: "To find out the 5 times table we make half of the 10 times table". In spite of this, when the teacher requested the students to write the computation strategy in mathematical language, some of them still used particular cases. As in the first task, the teacher stresses that she is not asking for particular cases but for a computation strategy that can be applied to "any number". At that moment, some students suggest the use of the question mark at the beginning of the sentence, but a few other non numerical symbols are suggested as well. However students show many difficulties to represent "half of any number", suggesting some creative ways to do it, like the following example shows, where they draw the half of one flower:

Figure 5 - One attempt to express the generalisation in symbolic language.

At that point, the teacher focused the students' attention on the exploration of the particular case presented in the statement of task, 36x5, revising the strategy previously discussed.

Teacher: We had 36x5... What did we do?

Rita: 35 times 10.

Teacher: And then?

Rita: We divided by 2.

Teacher: So, how can we do it? Any number times 5 is equal to...

Fábio: Is equal to half.

Teacher: What did we do first?

Gonçalo: We multiply by 10.

Teacher: We multiply by 10... But is 36x5=36x10 correct or not?

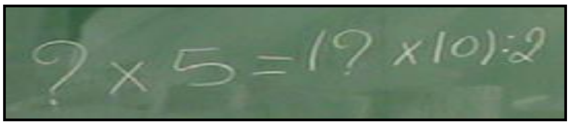
Students: No.

Fábio: No, then we made the half.

Teacher: And how can we represent half of it?

Rita: Dividing by 2.

Through the written expression 36x5 = (36x10):2, easily and naturally, students express the computation strategy in a general form. In that way, many students were able to express the generalisation of the strategy in symbolic language using different symbols like the ones shown below:



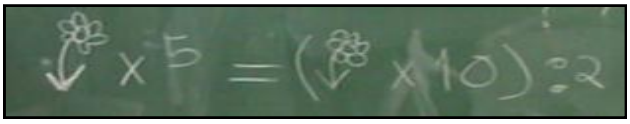


Figure 6 – Two expressions of the generalisation in mathematical language.

FINAL REMARKS

The analysis of these tasks exploration allows us to conclude that the students used computation strategies applied on particular examples to make generalisations. In this sense, students used those particular examples of the computation strategies in the meaning of quasi-variable (Fujii, 2003) as they were able to generalise beyond that, even if in the restricted context of the multiplication tables.

In the first task analysed in this paper, students were able to easily express the generalisation of the computation strategy in natural language. The numerical expressions presented in the task are looked at as quasi-variables, to foster the understanding of the underlying structure of the computation strategy. The expression of generalisation in natural language and the beginning of the translation to mathematical language was made collectively, from the contributions of different students. The introduction of the symbolic language arises from the suggestion of one student to use the question mark symbol. This symbol had already been presented in a previous task like an unknown number, but, in this task, the symbol appears as "any number", which represents a more complex idea that can provide a gateway to the concept of variable (Fujii, 2003).

In the second task analysed, students also easily express, the generalised in natural language. However, the expression of the generalisation in mathematical language was much more difficult for them as it requires the representation of "half of any number". Students showed some creativity by using some figurative symbols. This demonstrates that students are still at the beginning of the construction of their symbol sense. In spite of this, when the numerical strategy was taken up, students were able to express the generalisation in symbolic mathematical language.

In the Portuguese context, with the implementation of a new mathematics curriculum which assumes the importance of algebraic thinking in early years of schooling, this study shows an alternative way to the traditional approach to arithmetic. By focusing students' attention in the numerical relationships and in the study of proprieties of numbers and operations, they start to deal with algebraic ideas, developing also a deeper understanding of arithmetic. In this perspective, the use of quasi-variable thinking (Fujii, 2003), constitutes a potential pathway for algebraic thinking since it provides opportunities to work with several layers of awareness of generalisation (Britt & Irwin, 2011), starting from an arithmetical context familiar to students, as in this study.

The construction of generalisation as a collective process (Ellis, 2011) also stresses the importance of the reflective discourse in the classroom which attends to the individual contributions that support and sustain collective reflection (Cobb et al., 1997). Assuming the importance of the interaction between the teacher and the students, and between students, the collective discussions that promote a reflective attitude become crucial moments for students to learn from each other. The excerpts presented show how the students develop their understanding in those collective moments, and how the construction of generalisation was not just a product of one student, but the result of the class engagement.

Although the results reported in this paper concern an early stage of the teaching experiment, the latter development of the research (which is beyond the scope of this communication) shows how these initial tasks contributed to the development of the students' ability to generalise and to use meaningfully symbolic mathematical language.

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[1] This communication is supported by national funds through FCT - Fundação para a Ciência e Tecnologia - in the frame of the Project Professional Practices of Mathematics Teachers (contract PTDC/CPE-CED/098931/2008).

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