A FUNCTIONAL PERSPECTIVE ON THE TEACHING OF ALGEBRA: CURRENT CHALLENGES AND THE CONTRIBUTION OF TECHNOLOGY

Jean-Baptiste Lagrange

LDAR, University Paris Diderot, France

From the early nineties, most reformed curricula at upper secondary level chose to give functions a major position. The goal of this paper is to introduce key challenges resulting from this choice and to discuss the contribution that software environments associating dynamic geometry and algebra can bring to the teaching learning of functions. Two examples of situations based on the use of the Casyopée environment are proposed. They illustrate how educational design can handle key questions: experiencing covariation and using references to body activity is crucial for students' understanding of functions; making sense of the independent variable is a major difficulty that needs to be addressed by special situations; and understanding the structure of the algebraic formula in a function is critical.

INTRODUCTION

The functional perspective on the teaching of algebra is seen by curricula reformers as an effective approach to consolidate post middle school students' algebraic knowledge and to prepare them undertaking calculus. The use of technology, especially graphical and dynamic geometry software, is encouraged in an exploratory approach to functions. However, the topic of functions is complex and Kieran (2007, p. 710) notes that the reform gives way to "hybrid versions of programs of study that … can create additional difficulties for algebra learners". She depicts these programs of study as oriented toward the solving of realistic problems and towards multirepresentational activity, with the aid of technological tools, allowing for an algebraic content that is less manipulation oriented and a shift away from the traditional skills of algebra. Among the objections Kieran raises, I am particularly sensitive to the strong presumption, that, in these programs, symbolic forms will be interpreted graphically, rather than dealt with, technology being used to insist on screen (graphical) interpretations of functions. Clearly, there is a risk that, through these programs, students will have no access to understanding symbolic forms which are at the core of algebra and will be deprived of the power they offer for solving problems and more generally "for understanding the world". In a first part of the paper I report on the work of the Casyopée research group that for more than ten years worked in France in order to promote an approach to functions encouraging students' multirepresentational exploration together with work on algebraic symbolism. An example will be provided, highlighting students' achievements and difficulties. This will introduce a discussion in view of relevant literature and ideas for the design of lessons about functions. Another example will illustrate these ideas and the conclusion will revisit Kieran's objection.

OFFERING POSSIBILITIES FOR LEARNERS AND TEACHERS

The genesis of the Casyopée[1] group was in the years 1995-2000 when researchers from the University Paris-Diderot worked with teachers at the French experimentation of DERIVE and of the TI92 calculator. In a second period (2000-2006) the group was concerned by the instrumental difficulties and epistemological problems inherent to Computer Algebra Software (CAS) designed for advanced users and started to build a CAS tool that could be really used in the classroom. A central aim was to ensure consistency with current notations and practices at secondary level. We wanted also to avoid any command language by designing a menu and button driven interface like in Dynamic Geometry, because keywords are always difficult to handle for beginners and create confusions with mathematical notations. These choices helped to create an innovative algebraic tool contributing to a better appreciation of CAS by teachers. The group saw the potential of this tool for students to explore and solve problems involving modelling geometrical dependencies, for instance an area against a length. However the group was concerned that geometrical exploration and modelling had to be done separately from the work with Casyopée.

In the years 2006-2009, the group was involved in the ReMath[2] project that focused on multi-representation of mathematical objects. This was an opportunity to extend the representations in Casyopéee by adding a dynamic geometry window and representations of measures and of their covaration. This extension enabled covariations between couples of magnitudes to be explored and couples that are in functional dependency to be exported into the symbolic window. The outcome of this exportation is an algebraic function modelling the dependency, likely to be treated with all the available tools. In order to help students in modelling dependencies, this exportation can be done automatically (Lagrange 2010). We will refer to this functionality as "automatic modelling" below. After the ReMath project, the group worked to build a conceptual framework about functions and algebra (Lagrange & Artigue 2009). It is based on the idea that students approach the notion of functions by working on dependencies at three levels (1) activity in a physical system where dependencies are "sensually" experienced; (2) activity on magnitudes, expected to provide a fruitful domain that enhances the consideration of functions as models of physical dependencies; (3) activity on mathematical functions, with formulas, graphs, tables and other possible algebraic representations. The example of the next section will explain these choices.

THE SHORTER DISTANCE TO A PARABOLA

This lesson was carried out and observed in order to investigate 10th grade students' knowledge about functions and the way a tool like Casyopée can support developing this knowledge. The text of the problem was: M is a point on the parabola representing $x \to x^2$; the goal is to find position(s) of M as close as possible to A. The task was split into three subtasks corresponding to the levels of activity on functional dependencies outlined above: (1) make a dynamic geometry figure and explore (2) use the software to propose a function modelling the problem (3) use this function to approach a solution. I analyze how, in a class, the three subtasks were performed in a one hour session. For each subtask, a table displays in the left column an extract of an interaction between the teacher and one student (subtasks 1 and 3) or with the class (subtask 2) and the analysis in the right column.

Subtask 1. Understanding the problem, exploring

Interaction between the teacher (T.) and a student (S.)	*Analysis*
T. What is the problem? S. We have to find a place on the curve in order that M is as close as possible to A...Actually, we have to find a position of M in order that AM is minimum... T. How could we use the software? ... T. What could we ask him to calculate? S. Uhm... a calculation ...AM.	Passing from "M as close as possible to A" to "a position of M in order that AM is minimum" is a transition from the geometrical world to a quantification by a measure.

Subtask 2. Building a function

Interaction between the teacher (T.) and the student (S.) in a classroom discussion	*Analysis*
T. ...in order to get a better approximation, we need to define a function whose value is AM... but depending on what... S. on M... T. M is not a variable... When you move M, it depends on what? What gives the position of a point? S. The coordinates T. The coordinates... that is? S. x-coordinate and y-coordinate T. I have to choose, which one? S. y-coordinate T. The y-coordinate?... then... if I have to locate a point on the curve, what should you give to get the right position? S. The y-coordinate T. If you ask me for the y-coordinate 4... S. There are two points... we need to give the x-coordinate. T. With the x-coordinate, is it correct? S. Yes, we tried with the software, yM does	The students identified AM as a dependent variable in the preceding subtask. The challenge is to identify an independent variable and first to think of a magnitude that could be in covariation with AM. This is not obvious because the students perceive that the distance varies with M but do not understand how the position of a point can be quantified. They also do not understand that, because M is tied to a curve, only one coordinate is sufficient to quantify the position. For more advanced students, the parameterization of a point on a curve by the x-coordinate is obvious, but when students have to choose one, the y-coordinate is more appealing because the parabola is nearly vertical near the minimum and thus the y-coordinate seems more in relationship with the position. The teacher has to draw attention tothe

not work, xM does work. T. Yes, if you say, the point is on the curve, and I know the x-coordinate, then I know the position of the point...Then you can characterize the position by the x-coordinate.	fact that two points have the same y-coordinate. Students say that they tried both coordinates with the software and the discussion confirms the feedback of Casyopée.

Subtask 3. Using the function and reflecting on a solution

Interaction between the teacher (T.) and a student (S.) after he obtained an algebraic function by way of the automatic modelling	*Analysis*
T. ...this is the function... S... it is a monster… There is a square root and... T. Do you know why? S. Because it is a distance. T. How does it help you? Look at values on the graph... S. Yes it is easier to locate the minimum... T. Could you do a small report, how you get the function... and how it helped you.... S. Yes the variable and the image...	The student is surprised by the formula (the square root of a quadratic polynomial), but makes a connection with his knowledge about distances. However the formula is not, at this stage, a tool for a solution. At this stage, students learn to read a graph, and to coordinate this reading with a functional understanding.

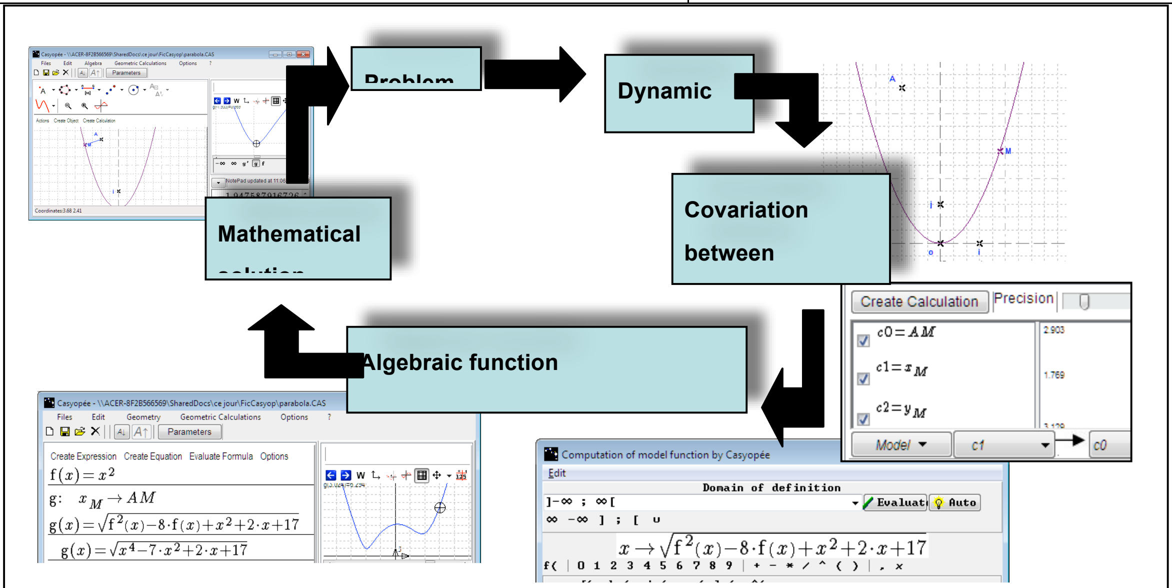

Figure 1: The shorter distance to a parabola: cycle of modelling

Figure 1 summarizes the different steps in the “cycle of modelling” underlying the minimum distance problem, and illustrates the specificities of the software. Subtask 1 above corresponds to passing from the problem to the exploration of a dynamic figure; subtask 2 corresponds to expressing a dependency between magnitudes;

passing from a covariation between magnitudes to an algebraic function is done thanks to "automatic modelling", and subtask 3 corresponds to the exploration of the function towards a mathematical solution.

TEACHING/LEARNING ABOUT FUNCTIONS, KEY QUESTIONS

Experiments of lessons like that in the preceding section draw attention to questions related to a functional approach to algebra. First, what is covariation, what role can it play in a functional approach and how did this idea appear? Second, how to characterize students' difficulties with the notion of independent variable? And third, what is at stake with the symbolism now that solutions of problems can be approached graphically or numerically thanks to the computer? In this section I will draw from math education research literature to get insight into these questions.

From process-object to covariation and embodied cognition

From the early nineties most of the studies concerning students' conception of functions were based on the distinction between the two major stances that students adopt towards functions: the *process view* and the *object view* (Sfard, 1991). Elaborating further the process–object duality in students' understanding of functions, mathematics educators suggested that students' understanding of functions can be considered as moving from an initial focus on actions and processes to more object-oriented views characterized by a gradual focus on structure, incorporation of properties and reification of mathematical objects. In this vein, from the middle nineties, a number of approaches developed to describe object-oriented views of function emphasized the covariation aspect of function (Thompson, 1994). The essence of a covariation view is related to the understanding of the manner in which dependent and independent variables change as well as the coordination between these changes. Covariational reasoning consists in coordinating two varying quantities while attending to the ways in which they change in relation to each other. This involves a shift in understanding an expression from a single input-output view to a more dynamic way which can be described "as 'running through' a continuum of numbers, letting an expression evaluate itself (very rapidly!) at each number" (Thompson, 1994, p. 26). However, this dynamic conception of variation seems not to be obvious for the students since it is essential to take into account simultaneous variation between magnitudes at different levels emerging in an ordered succession. Furthermore there is a need for situations that provide students with opportunities to think about the covariational nature of functions in modelling dynamic events.

The classroom situation above illustrates how exploring and modelling dependencies with the help of a tool can help students get a sense of covariation. Researchers like Rasmussen et al. (2004) and Botzer et Yerushalmy (2008) refer to embodied cognition to characterise the sense of a mathematical notion that students can get via interaction with a physical device. A central assumption of embodied cognition is that students' reference to bodily activity in physical settings and to emotions experienced in this activity, can be a basis for deeper understanding of calculus notions, as compared to a pure formal approach of these notions. Rasmussen et al. (2004) give the example of a university student who knew the formal definition of acceleration, but did not fully understand this notion. Experimenting with a rotating unbalanced wheel she identified herself with the wheel and became "friend with the acceleration".

Understanding the idea of independent variable

A particular difficulty in understanding functions deals with the idea of independent variable. Thompson (1994, p. 6) reports students' persistent 'mal-formed concept images (…) showing up in the strangest places". He particularly indicates that the predominant image evoked for students by the word 'function' involves two disconnected/separated expressions linked by the equal sign. Aiming to indicate students' difficulty in developing a conceptual understanding of the symbolic expression of functional relations and the role of particular symbols in it, he reports an example of a formula for the sum $S_n = 1^2 + 2^2 + ... + n^2$given by a student on the blackboard as a response to the teacher's request. The student wrote $f(x) = \frac{n(n+1)(2n+1)}{6}$ and none of the students found something wrong with this expression since it seemed to fit their image of function at that time. Here the student's use of symbols for the expression of a functional relation indicates an implicit consideration of it as a "template" consisting of two distinct parts in which the first one is used as a label for the second without linking at the conceptual level these two parts and the existing objects/quantities.

In the classroom situation of the preceding section the understanding of the independent variable is at stake: automatic modelling helps students to concentrate on the constituting elements of a function rather than on the production of a formula.

The role of symbolism

There is evidence in literature that the symbolism of functions is a major difficulty for students. Students' view of symbolic expressions can be of a pure input-output correspondence. In other circumstances, it can be pseudo-structural, the expressions being understood as an object in itself, not connected to functional understanding (Sfard, 1991). Slavit (1997) indicates the critical role of symbolism "confronted in very different forms (such as graphs and equations)" (p. 277) in the development of the function concept and suggests the need for students' investigation of algebraic and functional ideas in different contexts such as the geometric one. Even when students have access to basic proficiencies in algebraic symbolism, coordinating these proficiencies with an understanding of the structure of the algebraic formula in a function is critical and is particularly at stake when the function comes from a problem context. Most students fail in this coordination. Evidence of failure is given in the context of equation. For instance, van der Kooij (2010, p.122) notes that most students in a vocational high school "were able to do calculation on the pendulum equation $T = 2\pi\sqrt{\frac{1}{g}}$ while they gave no sense to an "abstract" equation $y = 2\sqrt{x}$". Kieran (2007) reports on low achievement across countries for items of a TIMSS survey involving production or interpretation of formulas to describe a phenomenon depending on a variable number.

In the above situation, symbolism is not at stake as a tool for solving. However, it is important in the students' understanding of the function that Casyopée displays the formula. The situation analysed in the next section, and designed for more advanced students will show how Casyopée can help reconcile symbolic forms and dynamic manipulation of mathematical objects and relationships.

THE AMUSEMENT PARK RIDE: FUNCTIONAL MODELLING AND DIFFERENTIABILITY

This classroom situation was designed to take up two challenges. The first one was the necessity for students to consider "irregular" functions before entering the university level because situations of modelling dependencies most of the time deal with infinitely differentiable functions not questioning the understanding of irregularities like discontinuities or non differentiability. The second challenge was to test with Casyopée the above mentioned embodied cognition assumption relative to the role of bodily activity in physical settings. More precisely here the situation was designed in order that students connect properties of irregular functions with a sensual experience of movements, in order to get a deeper understanding.

The problem was the following: a wheel rotates with uniform motion around its horizontal axis. A rope is attached at a point on the circumference and passes through a fixed guide. A car is hanging at the other end. The motion is chosen in order that a person placed in the car feel differently the transition at high and low point. It was expected that students would identify the difference, associate this with different properties of the function (non-differentiability and differentiability) after modelling the movement. The modelling cycle is similar to the above minimum distance problem, except for two points. (1) The problem is given in "real life" settings, the students being able to manipulate a scaled device, and then the first step of modelling consists in building a dynamic geometry figure replicating the device. The following indications are given to the students: the rope is attached to the wheel in a mobile point M and the guide is on the fixed point P. The car is in N (figure 2).

Figure 2

The wheel is supposed to be put into rotation by pulling on a horizontal rope jA. This implies not trivial constructions for the point M in order that the circular distance IM is equal to the linear distance Aj, and for the point N in order that MP+PN is constant. (2) More focus is put on the algebraic formula of the function. The students have to use Casyopée to get the derivative and should notice and identify precisely the points

t the lower point,
here is a drop shot
of non differentiability. The lesson was carried out with a 12th grade class in a 90 minute session. I report on this situation in five steps: (1) the students' spontaneous model of the physical situation (2) how they built a dynamic geometry model (3) how they chose the dependant and independent variables and how they interpreted this choice (4) how they worked on the algebraic function obtained via Casyopee's automatic modelling (5) how their understanding of the physical situation progressed after working on the algebraic function.

Students' spontaneous model of the physical situation

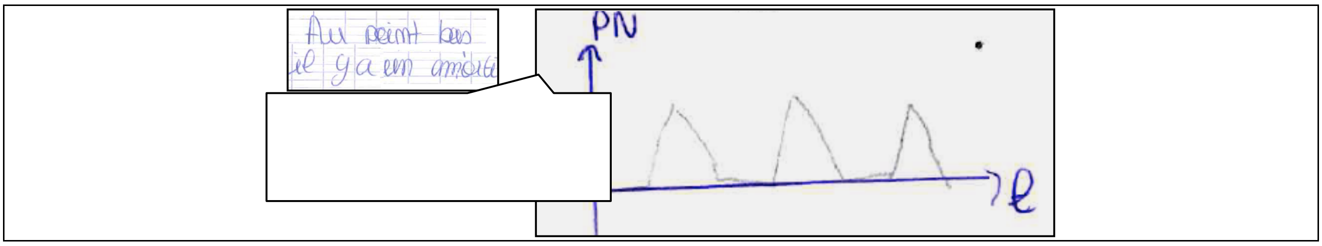

Figure 3: Students' spontaneous model

Starting the session and demonstrating by animating the scaled model, the teacher asked the students to describe what is happening at the lower point and whether it is different as compared to the high point. Figure 3 illustrates a typical answer. Students said that at the high point, the car stops and they had some difficulties explaining what was happening at the lower point. The more common expression, drop shot, is not accurate because it means that the car is arriving at a certain speed, stops and starts up again at a lower speed. Students illustrated by a graph of a piecewise linear function. Actually they thought that because the wheel rotates uniformly, the car's movement should be piecewise uniform.

Building a dynamic geometry model

This is a difficult part. Students' poor practical knowledge in trigonometry explains why they needed help to define M in order that the circular distance IM equalled the linear distance Aj. It seems more surprising that they found it difficult to define N in order to make MP+PN=2 (the length of the rope). After the teacher indicated that PN is known when MP is known, some students used a circle centred in P with a radius of 2-MP and defined N as an intersection point with the y-axis, and others directly defined N with the coordinates (0; y_P-(2-MP)).

Choosing the dependant and independent variables

Generally the students had no difficulties in operating the choice with the software. However, their expression was sometimes confused when explaining the choice. For instance a pair of students wrote in the report: "We choose distance Aj as the (independent) variable" and added "Aj is a function of the coordinates of N".

Working on the algebraic function obtained via casyopee's automatic modelling

The students obtained the derivative by using Casyopée under the form $x \rightarrow \frac{-\cos x}{\sqrt{2 \cdot \sin x+2}}$. Casyopée issued warnings because this function is not defined everywhere. Students ignored the warnings and obtained a graph with wrong vertical segments (figure 4). The teacher drew students' attention to these segments and students recognised that there should be discontinuities of the derivative corresponding to the low points. The teacher asked them to compute the position of these discontinuities. No students did this from the formal definition of the derivative. They rather came back to the physical device, looking for the value of Aj corresponding to the lower point of the car. After they found these values and excluded them from the definition of the derivative, they got a correct graph (figure 5).

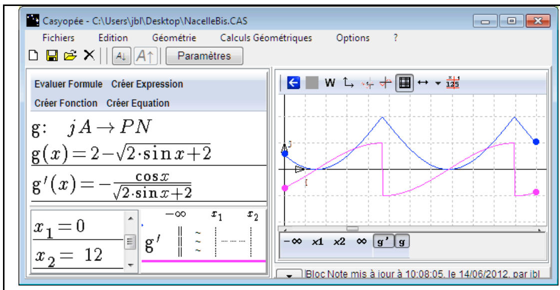

Figure 4: A graph of the derivative with wrong vertical segments

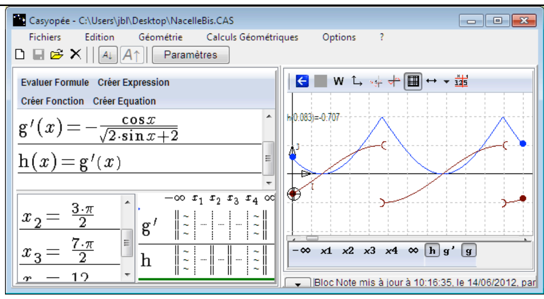

Figure 5: A correct graph of the derivative

Students' understanding of the situation after working on the algebraic function

Students' understanding was much better after working on the algebraic function. They identified the derivative and the car's speed, saying that the speed is null at the high point corresponding to a horizontal tangent on the graph of the movement. Implicitly, they recognised that at the lower point the car starts up again briskly at the same speed, speaking of "rebound" corresponding to non differentiability points, rather than of "drop shot" implying softer stop and restart.

CONCLUSION

Two examples have been developed in the paper. They helped to identify key questions about the teaching and learning of functions and to illustrate how educational design can handle these questions: experiencing covariation and using references to bodily activity is crucial for students' understanding of functions, making sense of the independent variable is a major difficulty that need to be addressed by special situations, and understanding of the structure of the algebraic formula in a function is critical. This latter point is in line with Kieran's concern mentioned in the introduction that, in recent curricula, symbolic forms will be interpreted graphically, rather than dealt with. In the two examples, students have access to the symbolic forms of the function at stake and are able to establish links with the magnitudes whose covariation the function models. It is a distinct design feature of Casyopée, and of the associated situations of use, to deal with symbolism, aiming to reconcile symbolic forms and dynamic manipulation of mathematical objects and relationships. In this approach the graphical and numerical settings are subsidiary means for exploration, which is different from current approaches that tend to strongly rely on these settings for problem solving. This is at least the beginning of an answer to Kieran's objection.

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[1] Casyopée, is an acronym for « Calcul Symbolique Offrant des Possibilités à l'Elève et l'Enseignant »

[2] "Representing Mathematics with Digital Media", 6th FP, IST-4-26751-STP, 2005-2009 (http://remath.cti.gr)

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