HIDDEN DIFFERENCES IN TEACHERS' APPROACH TO ALGEBRA – a comparative case study of two lessons.

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Algebra is a multi-dimensional content of school mathematics and can be approached in many ways. A fine-grained analysis was made of two lessons on introduction of variables in two Swedish classes with teachers who followed the same curriculum. When comparing the two lessons differences were found in the approach to algebra and the meaning of variable in what at first seemed to be lessons about the same content. Findings indicate that teachers shape the opportunities of learning by the approach they take.

INTRODUCTION

Introduction to algebra and the transition from arithmetic to algebra are known to be problematic topics in school mathematics, and recent research claim that these problems are more related to learning conditions than to cognitive limitations (e.g. Cai & Knuth, 2011; Kaput, Carraher, & Blanton, 2008). Algebra is a large content area and can be approached from many different perspectives. There is a considerable variation between different countries and educational jurisdictions concerning differences in school structure, as well as approaches to algebra and content for algebra teaching, and naturally these differences may lead to differences in learning outcomes (Kendal & Stacey, 2004). Differences found in curricula documents and textbooks can make comparisons of teaching and learning algebra difficult due to a lack of consensus about how to define algebra and algebra teaching. Research on teachers' use of curriculum (Remillard, 2005) has highlighted the teachers role in the enactment and meaning making of curriculum. The enacted curriculum is influenced by variables such as teacher's mathematical knowledge, beliefs, goals and traditions. Some such variables may be clearly outspoken, others may stay hidden and surface only in the enacted curriculum. The question raised in this comparative case study is whether two teachers who follow the same curriculum and textbook teach the same algebra, or if differences can be found on a subtle classroom level that may influence students' opportunities for learning. A fine-grained analysis of the given instruction shows that two different approaches to algebra are made although both teachers follow the same national curricula syllabus and plan the lesson influenced by the same textbook task.

DIFFERENT APPROACHES TO SCHOOL ALGEBRA

The discussion of how to define algebra is as multifaceted as are people's experiences of algebra and views on how to teach algebra. In the 1990's reports showed that school algebra was predominantly rule based and procedural (Kieran, 1992) whereas researchers suggested a broader approach including a generalisation, modelling, problem-solving and functional perspective (Bednarz, Kieran, & Lee, 1996). Today algebra is conceived as a branch of mathematics that deals with symbolizing general numerical relationships and mathematical structures. The learning of algebra can thus be seen as both *learning to see and reason* about relationships and structures, and *learning the formal symbolic language* used to express these relationships and structures. The former is often called algebraic reasoning and is prominent in literature about early algebra (Cai & Knuth, 2011; Kaput, et al., 2008). According to a characterisation made by Kieran (2004) school algebra is constituted by three types of activities; *generational activities* involving the forming of expressions and equations using variables and unknowns, *transformational activities* which are procedures such as simplifying, substituting or solving equations, and *global meta-level activities* where algebra is used as a tool, including problem solving, modelling, noticing structure or change and analysing relationships. When algebra is approached as a language (Rojano, 1996) generational and transformational activities dominate instruction, whereas a generalization (Mason, 1996) or a problem-solving (Bednarz & Janvier, 1996) approach to algebra involves more global meta-level activities. The current state of school algebra according to Kieran (2007) varies from country to country but tend to follow either a traditional or reform-oriented program. A traditional program has a strong symbolic orientation and approaches algebra as a language, whereas a reformist program deals more with functional situations through modelling and problem-solving activities. Using the term *curriculum* to describe the resources and guides used by teachers (Remillard, 2005), this study investigates if two lessons following the same curriculum give the same perspective on algebra or if different approaches to algebra surface through the teachers' transposition of the curricula documents.

METHOD

As part of data collected in an international comparative video study called VIDEOMAT (Kilhamn & Röj-Lindberg, 2012) two Swedish grade 6 classrooms were videotaped during a lesson on introduction to the concept of variable. Data was collected *in situ* using three cameras during a sequence of four lessons when the teachers planned to introduce algebra. Both schools were public schools, following the national curriculum (Lgr11) and using the same textbook (Carlsson, Liljegren, & Picetti, 2004). Both teachers were trained as generalist teachers for grades 1-7 with approximately 10 years of teaching experience. Explorative analyses were made of the videos and of verbatim transcripts (in Swedish and translated into English). Similarities and differences of the two lessons were sought, particularly in relation to different approaches to algebra as described in research literature.

CURRICULA DOCUMENTS AND TEXTBOOK TASK

Before presenting the classroom analysis the curriculum that serves as a common point of departure for both teachers is described. When planning the algebra unit both teachers refer to the National Curriculum (Lgr11) and the textbook (Carlsson, et al., 2004) including its teacher guide as the foundation of their instruction. Lgr11 includes a mathematics syllabus consisting of general aims and core content in mathematics (Lgr11, pp. 59-46). Algebra is a core content, including for grades 4-6:

> "Unknown numbers and their properties and also situations where there is a need to represent an unknown number by a symbol; simple algebraic expressions and equations in situations that are relevant for pupils; methods of solving simple equations; and how patterns in number sequences and geometrical patterns can be constructed, described and expressed."

There is a specific algebra unit in the textbook used by both teachers. The unit starts on pages 94-99 dealing with the meaning of the equality sign and simple equations with one unknown. On page 100, labelled *'Variables can vary',* the concept of variable is introduced using age differences with the following information given (all translations made by the author of this paper):

> "We can call Amer's age *a*. Sama is 4 years older than Amer. That makes Sama's age $a + 4$. The value of *a* changes when Amer's age changes. The value of *a* can vary, *a* is a variable." (Carlsson, et al., 2004, p. 100).

This introduction is followed by a task on the same topic (fig 1), and is commented in the teacher guide in a short paragraph:

> "The word variable is another new word for the students. A variable is a quantity that can vary. On page 100 we have ages as an example of variables. All students understand that ages vary – when my sibling is one year older I am one year older. To work with variables is very useful in mathematics, for example all formulas are based on the fact that you can vary the value of the variable. Formulas are valid for a range of values."

Osmond is 3 years older than Mohammed.
Leyla is 5 years younger than Mohammed.

How old is Osman when Mohammed is
a) 10 years b) 15 years c) 30 years

How old is Leyla when Mohammed is
a) 10 years b) 15 years c) 30 years

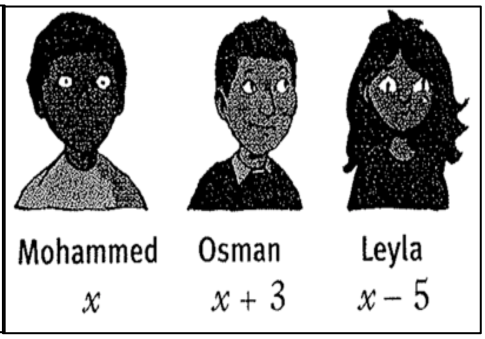

Figure 1: First task on introduction to variables (Carlsson, et al., 2004, p. 100)

The description of the *algebra* and *variable* given in the curriculum follow a more traditional that reformist-oriented program for school algebra.

RESULTS

The results presented here focus similarities and differences concerning the content matter taught by two teachers, Ms B and Ms C, in two grade 6 classes in two Swedish schools. Undoubtedly there are demographic differences as well as differences in classroom organization that have an impact on learning outcomes which are not in focus in this study, and no comparisons of actual learning outcomes is possible from the collected data. Instead the intent is to look closely at the art of the content matter taught and the approach taken to algebra in each lesson, highlighting similarities and differences in the first classroom activity in the lesson on *introducing variables* that surfaced in the anaysis.

Ms B's lesson

Ms B starts her lesson on introducing variables by referring back to tasks they have worked with using a variety of symbols to stand in place of a number, and asks the students to find a more simple, a more convenient way of writing the mathematical statement "some number added to 2". Ms B emphasises many times that the point of algebra is to write something in a simple and quick way. "And maths is very much about actually finding convenient ways of doing things" The term *variable* is said to be related to *varying* illustrated by the statement that x varies in $x+2=5$ and $x+2=7$ because it does not represent the same value.

The age relation task from the textbook (fig 1) is projected onto the whiteboard and students are asked to work with it in small groups. Later they will show their solutions using a document camera and interactive whiteboard so that they can be discussed in class. Ms B particularly points out that she expects them to be able to say how they worked out the answer. "I want you to fill in how you have reasoned using a variable. You don't need to erase, rather you add that, the way you reasoned" and she points out that they need to know what x means "what is it that you– in the problem so to speak what is it that you have found out? It might be good to know then what, this symbolises. Because x is a symbol for something."

When Ms B discusses the students' solutions in class, much time is spent going back and forth between different representations; words ⇔ symbols. On the whiteboard Ms B first shows answers without any variable, asking if the calculations are correct, which they all agree to. One example of such an answer is:

> a) 13 years. Osman is 3 years older than Mohammed so when Mohammed is 10 years Osman is 13 years.

Then Ms B highlights answers where x is present in the explanation, ending with the student focus group (FG) who wrote:

> a) $10+x+3=13$
> $15+x+3=18$
> $30+x+3=33$ $x=0$ Osman is always 3 years older than Mohammed.

In this example the students have added Mohammed's age and Osman's age ($x+3$) and the difficulty does not lie in getting the correct answer but in understanding the meaning of x. Excerpt 1 shows how the group discussed the problem. One student has solved the problem of finding the sought ages straight away [1], but then the group spends another 10 minutes discussing how to write it down. They include an x because they know it is supposed to be there [2]. They try adding the symbolic expressions of the ages of Osman and Mohammed [2 - 4], but the discussion of what x symbolises continues until S3 finally suggests that $x=0$ [5].

Excerpt 1: FG discussing the age task, extracts from a 10 minutes long interaction.

[1]	S1	it is just 13, 18 and 33
	S3	so we can start working out how old they are now, how old they are.
	S2	Osman is 3 years older than Mohammed
(…)		
[2]	S4	Shall I work it out in an algebra way?
	S1	no (protests and wants to go on)
	S4	okej. Well then this is what we do. That is x plus x plus 3.
(…)		
[3]	S4	(pointing to her paper where she has written 10 + x + 3) That is Osman's age and that is someone else's age. It is Mohammed's age
(…)		
[4]	S2	but look here. We know Osman is 3 years older than Mohammed all the time.
	S1	So it has to be 13
	S2	yes 18, and then 33
	S1	yes. I said so all along. That's how easy it is.
	S2	because I tried, we tried to work out how old they are now. But that is impossible.
	S4	(writes 10+x+3=13, 15+x+3=18, 33+x+3=33)
[5]	S3	but eh, how about the x?
	S4	but x, they, here it is an age
	S3	well but
	S4	it's one of them. It's 3
	S3	but you can't have just x there? If it's there you think it means something ( ) write something
	S4	write x plus 3 is equal to, eh, 3
(…)		
[6]	S3	(writes x=0) Do you think this is, do you think this is correct?
	S2	x is equal to zero?
	S3	yes because x is nothing it is just what his age is called.

In the following whole class discussion (excerpt 2), Ms B directs attention to the information given in the task, particularly the algebraic expressions under the picture. She wants the students to express the ages in the algebraic way; however, this is not easy since they all agree that the answer is already given in the text.

Excerpt 2: Extracts of Ms B's whole class discussion of the age task.

Ms B:	What information did you use, to arrive at what Osman was? What does Peter say?
Student:	How old the others were.
Ms B:	And how did you find that out? Marcus?
Student:	It's in the text.
Ms B:	It's in the text. There was some group who discussed something else on this page.
(…)	(Ms B returns to the task and points at the algebraic expressions under the picture)
Ms B:	Was there any group who explained the ages this way? When you tried to find out, how old Osman was

(…)
Student: Eh, we, we checked what the x:s meant and put it together.

Ms C's lesson

Ms C starts the lesson reminding the students of equations and introducing the term variable as being "reminiscent of variation for example. Thus it is something that varies". The introduction is centred on a description of the ages of Ms C's own family members, and ways of describing those ages in relation to her. She writes ages and age differences on the whiteboard. Ms C uses the first letter of each name to represent that person's age. Then she relates all the members' ages to her own, "describe our ages based on a variable then. I will describe it with a, a, with letters and numbers. And I will base it on myself all the time". When describing relations Ms C introduces *formulas* as illustrated in figure 2 and says:

> "Eh, for me to describe dad's age I'll take my age and then I'll add years, because he's older than me. This– my– since J means 36 right now. Mm, eh, so I add 27 there. That means that one can calculate via this formula, if one knows that I'm 36, then, that dad is 36 plus 27, which is 63. And we can also by looking at this understand how old my dad will be when I'm 40. When I'm 40, then the same formula holds, he's always 27 years older. Eh, so then you get 40 in there. How old is my dad when I'm 40, Alex?"

M= J+27	**M**ark	**A**nn-**C**hristin	**J**enny	**E**mma	**A**nna	**L**otta
L = J-13	63 yrs	60 yrs	36 yrs	32 yrs	29 yrs	23 yrs

Figure 2: What Ms C wrote on the board during the introduction.

By asking the students to calculate the father's age (as well as other members) at different points in time (when J is 36 and when J is 40), Ms C illustrates the idea of a varying variable. Students are then asked to describe their own families in a similar fashion. When the students have drawn their families and given them names they are asked to "write down a variable and start from your family. For example: mom's age, equals, my age, plus …" See figure 3 for an example of a student's work. Since they have all written several variables (one for each family member) what Ms C means is probably that they should decide which variable to relate to and then write down formulas that show the other members' ages in relation to the chosen one.

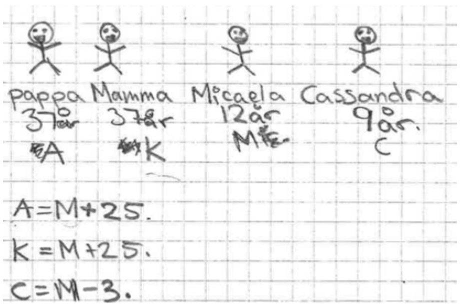

Figure 3: example of student work from Ms C's class.

During the following whole-class discussion of the student's results (see excerpt 3) Ms C asks: "What have you chosen for a variable", meaning which variable they have chosen as independent.

Excerpt 3: Extract from Ms C's whole class discussion on the family age relation task.

Ms C:	Then you'll tell us first about your family.
Student E:	Eh, okay, dad ( )
Ms C:	Dad, he is…
Student E:	Eh, 48.
Ms C:	48? Mm. Mom?
Student E:	Eh, 44
Ms C:	44 years old. And then you?
Student E:	Eh, I'm 12. (Ms C writes P 48, M 44, E 12).
Ms C:	You're 12. Eh, and what have you chosen for a variable?
Student E:	Eh…
Ms C:	Sh!
Student E:	I've chosen, A… Or, dad…
Ms C:	Yes.
Student E:	…equals me plus… should I say what it makes?

After some negotiation they work out 48 -12=36. Ms C writes on the board: P = E + 36. Here the independent variable is E (student's age) and the fathers age (P) is expressed by a formula using the variable E.

ANALYSIS OF SIMILARITIES AND DIFFERENCES

When comparing these two introductions to the concept of variable some similar features can be noticed:

- The concept of variable is introduced using age relations as suggested by the textbook, involving the use of symbolic language to represent age relations.
- A letter or other symbol for an unknown in an equation has been introduced before the concept of variable.
- Variable is described as something that varies and there is an underlying but unclear distinction between a specific unknown and a variable.

Some differences between the two lessons concerning the algebra content and the approach to algebra were found and will he be described in terms of i) the framing of the algebra task, ii) the meaning of algebra, and iii) the meaning of variable.

Framing of the age relation task

Ms B starts her lesson with group discussions based on the textbook task. In this task all relations are given both in words and symbolically and students are asked to interpret them to calculate ages in different scenarios. Only one variable is used. It is easy for the students to find the ages. The task gives the general case (the relation) and students are asked to calculate specific cases. Students interpret and try to understand the language of algebra. The activity is transformational.

Ms C asks her students to describe their own families. In that setting all specific ages are known and the task is to find a way of describing their relations, both with words and symbolically using a formula. More than one variable is used. It is a question of choosing an independent variable, of assigning symbols and of describing relations. The task gives specific cases and students are asked to express a generality. Students use the language of algebra to model a situation, and the activity could be characterised as generational and global meta-level.

The meaning of algebra

Ms B tells her students that algebra is a simpler, quicker, more convenient way of writing down mathematical statements. The task itself is easily solved without algebra so the introduction of algebra in the activity does not make it more simple or convenient (as shown in excerpt 1). The meaning of algebra conveyed in this lesson is "algebra as (an efficient) language". Students first solve the task by calculating specific values and then try to express what they already know in the new language.

Ms C introduces the concept of *formula* along with the term *variable* so that the age relations in the task can be modelled. Algebra is used as a tool to model age relations and as a result formulas can be used to predict different scenarios (my fathers age when I am 40). The meaning of algebra conveyed in this lesson is "algebra as generalization" and "algebra as a problem solving tool".

The meaning of variable:

In both lessons students are told that a variable varies. In Ms B's examples it is implicit that the variable stands for a range of values (Osman's age as x+3), but in the students' equations x does not vary (10+3=13; 10+x+3=13). Moreover, in Ms B's examples of a variable in the two equations x+2=5 and x+2=7, x represents an unknown number in each case rather than a range of values. Ms B pointed out that "x is a symbol for something", but in the students talk of the meaning of x it is unclear to them what x symbolises: "If it's there you think it means something", "x is nothing it is just what his age is called" (excerpt 1), and "we checked what the x:s meant and put it together" (excerpt 2). The meaning of variable in Ms B's classroom is a letter that symbolises something else and is consistent with an approach to algebra as a language and algebra as symbolic manipulation.

In Ms C's lesson students are asked to write formulas including two variables (e.g. M = J + 27), which results in equations that are true for a range of different values. There is some confusion as to what in the task is a variable: "write down a variable and start from your family. For example: moms age, equals, my age, plus…" When Ms C says "what have you chosen for a variable" (excerpt 3) the information she seeks is actually threefold: independent variable (E), dependent variable (P) and formula describing the relation (P = E + 36). For Ms C the variable is a point of departure, it is what you relate to in a formula. "describe our ages based on a variable then. I will describe it with a, a, with letters and numbers. And I will base it on myself all the time". The distinction between *variable* and *formula* is a bit fuzzy, but has situates the idea of variable in a context of expressing relations. The meaning of variable conveyed in Ms C's classroom in consistent with a functional approach to algebra where variables are used to model how things relate to each other.

DISCUSSION

The analyses made of the two lessons indicate that the two teachers approached algebra in different ways. Ms B approached algebra as if it were a foreign language, a language of symbols that students need to learn through activities of interpretation and translation. Eventually this new language will prove to be efficient and facilitate mathematics. The learning process involved interpreting a general statement in specific cases. Only one variable was used although implicitly there was a second, dependent variable. Ms C approached algebra as if it were a problem-solving tool, useful to model, generalise and express relations. The learning process involved taking a specific case to a general level. Several variables were used modelling one variable as a function of another and incorporating the concept of formula. In Ms B's lesson they *talked about* algebra as an efficient language, but in Ms C's lesson the students *used* algebra to reason about something. Perhaps this difference can be boiled down to the fact that Ms B introduced variables in expressions with only one variable, and Ms C introduced variables in expressions that were statements about relations between two variables.

Conjectures about differences that are made in this paper do not take into account what happened after the introductory lesson, it is not ruled out that a teacher may approach algebra differently at different points in time or consciously use different perspectives separately. The intention was to illustrate differences in what *may seem* to be similar introductions to the concept of variable based on the same curriculum. It can be argued that these differences may influence opportunities of learning offered in the lesson. It might well be that the teachers themselves are unaware of these differences and that their learning goals are the same. An implication of this study is that comparative research can be useful to detect hidden differences in teaching.

REFERENCES

Bednarz, N., & Janvier, B. (1996). Emergence and development of algebra as a problem-solving tool: continuities and discontinuities with arithmetic. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra. Perspectives for research and teaching.* (pp. 115-136). Dortrecht: Kluwer Academic Publishers.

Bednarz, N., Kieran, C., & Lee, L. (Eds.). (1996). *Approaches to algebra: Perspectives for research and teaching* (Vol. 18). Dortrecht: Kluwer Academic Publishers.

Cai, J., & Knuth, E. (Eds.). (2011). *Early algebraization. A global dialogue from multiple perspectives*. Berlin: Springer.

Carlsson, S., Liljegren, G., & Picetti, M. (2004). *Matte Direkt Borgen 6B*. Stockholm: Bonniers.

Kaput, J., Carraher, D., & Blanton, M. (Eds.). (2008). *Algebra in the early grades*. New York: Routledge.

Kendal, M., & Stacey, K. (2004). Algebra: a world of difference. In K. Stacey, H. Chick & M. Kendal (Eds.), *The future of the teaching and learning of algebra: the 12th ICMI study.* (pp. 329-346). Boston: Kluwer Academic Publishers.

Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York, USA: Macmillan Publishing Company.

Kieran, C. (2004). Algebraic thinking in the early grades: what is it? *The Mathematics Teacher, 8*(1), 139-151.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels. In F. Lester (Ed.), *Second handbook of reseach on mathematcis teaching and learning.* (Vol. 2, pp. 707-762). Charlotte, NC, USA: Information Age Publishing.

Kilhamn, C., & Röj-Lindberg, A.-S. (2012). Seeking hidden dimensions of algebra teaching through video analysis. In B. Grevholm (Ed.), *Nordic research in mathematics education, past, present and future*. Oslo: Cappelen Damm.

Lgr11. (2011). *Curriculum for the compulsory school, preschool class and the leisure-time centre 2011*. retrieved 20 July, 2012, from: http://www.skolverket.se/publications

Mason, J. (1996). Expressing generality and the roots of algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.), *The role of problems and problem solving in the development of algebra.* (pp. 65-86). Dortrecht: Kluwer Academic Publishers.

Remillard, J. (2005). Examining key concepts in research on teacher's use of mathematics curricula. *Review of Educational Research, 75*(2), 211-246.

Rojano, T. (1996). The role of problems and problem solving in the development of algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 52-62). Dortrecht: Kluwer Academic Publishers.