THE EQUAL SIGN AND ALGEBRAIC THINKING IN PRIMARY SCHOOL IN PORTUGAL

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ABSTRACT

In this paper we describe a research about some difficulties found by young Portugal children when using equal sign in numerical equalities, at the time when a new program was introduced in Portugal. We found that teachers still didn't introduce the general perspectives, just some examples. Some algebraic properties are used in schools, but the teachers are not aware of their relational meaning. Results are better than in other international studies, probably because of such introduction of relational examples. Nevertheless, the results still reveal difficulties on relational understanding about not only the use but the meaning of equivalence itself as well.

INTRODUCTION

The theme of this work arose from the understanding of the importance of the equal relationship in the development of algebraic thinking. The new primary school program of mathematics in Portugal stresses relations between numbers and between numbers and operations. Besides, it introduces explicitly the theme of algebraic thinking for primary students (DGIDC, 2007). Nevertheless, as mentioned by Carpenter, Franke and Levi (2003), many students throughout the primary school, think that the equal sign should be followed by an answer. They give an example: 18 + 27 = _ + 29 and many students would answer by writing 45 in the space. Falkner, Levi and Carpenter (1999) defend that the concept of equality is basic to the development of children algebraic thinking. In the beginning of their project they asked students to solve the problem: 8 + 4 = _ + 5. So, from the beginning, they could see that children saw the equal sign as an instruction 'to do', not as a symbol representing a relation therefore they had a procedural understanding of the equal sign.

Taking into account this situation, and considering that there is a new curriculum in Portugal in which algebraic thinking is explicitly included, we have initiated a study concerning the use of the equal sign by primary school children in Portugal to try to ascertain what is the situation there. Our main research question is *if, with the introduction of the new program, we can find some satisfactory results on questions involving the notion of equality as well as relational thinking*.

Since we could not find any study in Portugal on this issue, prior to the introduction of the aforementioned program, we chose to compare results with studies in other countries. Our study was implemented in a group of schools that had already followed the new program (not all schools in Portugal have begun at the same time) and the follow-on ‘Mathematics action plan’ that gave support for its implementation. This group of schools presents results in the national exams that are slightly above average (9% above average for the 4th year of schooling and 2% for the 6th year).

THE EQUAL SIGN AND THE EQUIVALENCE RELATION

An elaborate notion of equality that happens as a product of instruction is the notion of equivalence relation. It has to do with the reflexive, symmetric and transitive properties (Behr, Erlwanger and Nichols, 1980). Smith (2011) refers to equality as the acknowledgement by students that the equal sign represents the equivalence between sets. Equality implies some mathematical properties that assume especial importance on the understanding of some more advanced concepts in Algebra

PEDAGOGICAL ISSUES AND STUDENT DIFFICULTIES RELATED TO THE EQUALITY

The National Council of Teachers of Mathematics (2007) recommends that students, from the first years of schooling, should start working to understand the concept of equality. They mention that it is common students see the equal sign as meaning ‘we need to give the answer’. Ginsburg (1977, in Freiman and Lee, 2004) and Haylock (2006) among others establish a link between the equal sign and the operation, unable of imagining them apart. Haylock (2006) stresses that in algebra the equal sign must be seen as representing equivalence. However many students and some teachers use the sign for a sequencing of operations, which is inadequate.

We agree that the usually poor results are related to school practices and teacher’s interpretation of national programs. As an example of such influences, Ma (2009) describes a situation in which she questioned an American primary school teacher why she left her students use, for example, “3+3x4=12=15”. She answered that, as they performed the operations in the right order, and arrived to the right answer, she didn’t see any problem with it. Ma goes on to counter that for the Chinese teachers, it is inadmissible that two values placed in two sides of the equal sign can be different. Such comments are similar to those reflected by Schliemann, Carraher and Brizuela (2007) when asserting that students’ difficulties can be derived from the way they were taught during the first years of schooling, because the equal sign is usually introduced as directional operative with the meaning of ‘gives’ or ‘result’

In a study by Behr, Erlwanger and Nichols (1980), on children understanding of the equal sign, they found a strong propensity among children to accept the equal sign as admissible in a sentence, only when one or more operational signs precedes it. The authors claim that children saw the equal sign as an indicator of results instead of expressing an equality relation. Having in mind similar observations, Falkner, Levi and Carpenter (1999), in a study with children from 1st to 6th grades, presented the following equation 8 + 4 = □ +5 and asked children to complete it. The answers were gathered in a table 1. Less than 10% of the students in any schooling year gave a correct answer and surprisingly the performance did not improve with the progress through school. Primary school children apparently do not understand the equal sign as a relation and see it as a sign to perform the calculation before, with the result placed immediately after (Falkner, Levi and Carpenter, 1999).

Answers in percentage				
Schooling Year	7	12	17	12 e 17
1st and 2nd	5	58	13	8
3rd and 4th	9	49	25	10
5th and 6th	2	76	21	2

Table 1. Percentage of answers obtained in the equation 8 + 4 = □ + 5

In a study by Freiman and Lee (2004), the results have shown few mistakes in situations of the type 'a=a' or 'a+b=c'. Children mistakes occurred in the cases 'c=a+b' in which children repeated one of the values. In the case 'a+b=c+d', where three numbers are given and the fourth is requested, children's performance was more complex. When the missing number was 'c', some children placed the sum 'a+b', acting as if 'd' was not there. Other children ignored 'a' and performed 'c=b-d', for example '4+8=3+5' while others just repeated one of the terms *a*, *b* or *d*. in this study, the authors intentionally presented one of the equations (8+4=□+5) in Falkner, Levi and Carpenter (1999). It is clear that the results obtained on the higher levels were much different. A comparative table was built:

Falkner, Levi and Carpenter		Freiman & Lee	
School years	% of correct answers	% of correct answers	School years
1st and 2nd	5	3	Kindergarten
3rd and 4th	9	77	3rd
5th and 6th	2	86	6th

Table 2. Comparison of student answers in the equation 8 + 4 = □ + 5

THE EQUAL SIGN AND ALGEBRAIC THINKING

Recent research on children knowledge –based on the traditional sequence arithmetic-then-algebra - has shown specific obstacles to algebra learning related with the difficulties that computational arithmetic learning creates (Kaput, 2008). For example, the approaches limited to equality and the equal sign in arithmetic mentioned by Kieran (1992, in Kaput, 2008) are known to interfere with the later algebra learning (Fujii, 2003 in Kaput, 2008). It is not surprising that students develop this notion of equality (operational instead of relational) since many textbooks limit their tasks to an operational type (Smith, 2011).

We agree with Kieran (2004), that in the transition from arithmetic to algebra, students need to make adjustments in the way they think, including those quite competent in arithmetic. In the initial levels there is a strong emphasis on obtaining the answer instead of on the representation of relations. In the presence of an equal sign, students see it as a frontier between problem and solution with a left-right direction. Kieran suggested five ways to develop algebraic thinking: 1) focus on relations, not on calculating an answer; 2) focus on the inverse operations, not only on the operations themselves, and on the ideas of doing and undoing that are part of the process; 3) focus as much on the representations as on problem solving, not just on solving; 4) focus both on the letters and on the numbers, instead of only on the numbers, including working on letters that can be unknowns, variables and parameters; accept open literal expressions as answers and compare expressions for equivalence based on properties; refocus on the meanings of equal.

Following such a perspective, Molina, Castro and Ambrose (2006) have made work with 3rd grade students to promote algebraic thinking. They also tried to find how students understood the equal sign. Their results enable the verification that initially students presented difficulties similar to other authors and in particular, all students evidenced an operational interpretation of the equal sign. However, students progressively evolved to a relational interpretation.

We don’t know if something like this happened, when the National program introduced such perspectives, and what could happen if the teachers introduced these ideas in their classrooms. And, consequently, which are the influences on student’s answers in such a situation. That’s the interest of our empirical study.

EMPIRICAL STUDY

Considering the research presented above, we have constructed a task involving open numerical equalities. This task was given to an intentional sample of students from the 1st to the 6th graders of a group of schools at the end of the year 2010/2011. As to the selection of the students, we took all students from 6 different schools that were still in the school building after school time, that were in extra curricular activities. It’s usual in Portugal that some weeks after the end of the academic year, the schools are open to help some families with financial difficulties to have their children at home. Therefore, it's not a selection due to cognitive reasons, but just for socioeconomic reasons. No students from the third grade were in the sample because of this situation. These activities are not curricula bounded and are monitored by non-specialized staff. We decided to include to the sample a group of 6th year of schooling for which a whole class in the same group of schools was chosen randomly. Students had no time limitation to complete the task. Our final sample was constituted by 24, 20, 24 and 27 students, respectively from the 1st, 2nd, 4th and 6th years of schooling.

The task was constituted by eight open numerical equalities in which students had to place a number to make it true. The equalities were:

(a)	8 + 4 = □ + 5	(e)	6 + □ = 4 + 5
(b)	8 = 3 + □	(f)	□ = 4 + 3
(c)	17 = □ + 17	(g)	□ + 4 = 5 + 7
(d)	3 + 5 = 2 + □	(h)	9 + 7 = □ + 9

The items were similar to those used in other studies, but particularly selected to give the possibility that students consider (as solving strategies), some usual relational properties, apparently experienced in school practices. Item f is the regular operation in which we put the result at the beginning. With Item c we try to identify the possible influence of teachers' use of zero neutral property. With item (h) we want to see the possible influence of commutative property. The items (e) and (b) are related to the use of lacunar examples of type " __ + a = b " that are used regularly in schools. And examples (a), (d) and (g) are related to the possible use of the property a+ b = a+1 + (b-1); or the a+ b = (b-1) + (a+1) used sometimes in mental computation exercises. As we can see, just the first item was the same as in the studies of Falkner, Levi and Carpenter (1999) and Freiman and Lee (2004).

In our sample the results about item (a) were as follow in table 3:

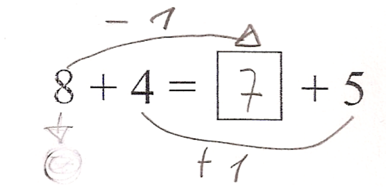
	Number of students	Answer 8 + 4 = 17 + 5	Answer 8 + 4 = 12 + 5	Other values	Answer 8 + 4 = 7 + 5
1st year	24	3 (12.5%)	13 (54.1%)	4 (16.7%)	4 (16.7%)
2nd year	20	1 (5%)	8 (40%)	1 (5%)	10 (50%)
4th year	24	1 (4,17%)	6 (25%)	3 (12.5%)	14 (58.3%)
6th year	27	1 (3,7%)	5 (18,4%)	1 (3.7%)	20 (74%)

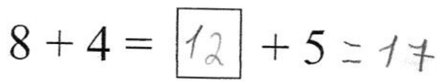
Table 3. Students results in the equality 8 + 4 = _ + 5

From table 3 we can see that 12.5% of the first year students added all numbers obtaining 17. In the other years, it happened too but in only one case per year. Some students added 8 and 4, disregarding number 5 on the other side of the equal sign. That was the case of 54.1% of first year students, decreasing with other years, attaining 18.4% in 6th year.

On the other hand, 16.7% of the first year students 50% of the 2nd, 58.3% of the 4th and 74% of the 6th answered correctly, establishing a relation between the two numeric expressions separated by the equal sign.

Below there is the resolution of one 4th year student where it is clear that he has established a relation, representing it by quantified arrows.



One 1st year student has solved it (incorrectly) like this: 

This student clearly felt that the expression was incomplete without the total sum and so appended a second equal sign. Clearly this student was not seeing the equal sign as establishing a relation between the two terms.

In the last equality, 9 + 7 = _ + 9, whose presentation was similar to the first considering that the empty space was just after the equal sign, yet quite different as the commutative property could be used, 16.8% of the 1st year answered correctly, 55% in the 2nd, 75% in the 4th and 81.5% in the 6th year of schooling. These results are slightly better than for the first equality, with much better results in the 4th year. Results are quite different when considering the equality 17 = _ + 17, with much better performance in all years of schooling. We really cannot ensure that this is because of the use of neutral property, but it could be, because it's traditionally introduced since first year of school.

	1st year	2nd year	4th year	6th year
8 + 4 = _ + 5	16.6	50	58.3	75
9 + 7 = _ + 9	16.8	55	75	81.5
17 = _ + 17	62.5	95	87.5	96.3

Table 4. Correct results in percentage for each year in three equalities selected

Concerning the remaining equalities in the task, that were not analysed here, in general the results were better. However, only in two questions did all the students of a given year answer correctly. That happened in the question related to 8=3+□, where all the 6th year of schooling answered correctly and for □=4+3, this time for the 2nd year students.

The best performance of the 1st year students was in □=4+3 with 79.2% and their worst performance was in 8+4=□+7 and 9+7=□+4 with 16.7%. The worst performance of the 2nd year students was in 8+4=□+7 with 50%. The best performance of the 4th year students was in 8=3+□ with 95.8% and the worst in 8+4=□+7, with 58.3%. The worst performance of the 6th year students was in 8+4=□+7 with 74%.

After summer, we asked teachers of such students about their previous work. We identify that the textbooks used still don't use enough relational perspectives, but introduced some examples as it's prescribed in the new curriculum. Three out of six teachers assume "it's not enough to use the equal sign as a result of an operation, as it's shown in the new curriculum" (teacher A). According to the comments of the teachers, they used some examples of equalities in which the equal sign is not related to a result of an operation. None of them asserts spontaeously the use of problems in an "equation form" but when they were required to show examples of finding missing values in arithmetic problems, they show us examples as ___ + 9 = 11, but a few examples like 4 + __ = 10 + 1 in which it appears a sum on the right side of equal sign and the missing value on the left side.

When we ask the teachers about the use of arithmetical properties, they talk about commutativity, or distributivity law, but none of them tell us about properties of equalities as " a+b = (a+1) + (b-1)" or "a+b= 2a + (b-a)". Even using mental computation exercices, in which some of these properties shoud be used, it's not clear for them the need for expliciting the variability relations used as a background.

CONCLUSION

First of all, our study offer Portuguese data about an interesting point necessary to help students when they start studying algebra, to overcome the idea of the sign of equal as an operator and acquire instead the idea that the sign of equal represent a mathematical relation (Smith, 2011). From our study, we see that Portuguese teachers are not aware of the relational meaning of properties and the corresponding variability use, even with new curricular influences.

It appears, even though the results are better than for the studies discussed, that it is of the utmost importance to promote some changes in teaching practices in Portugal towards the promotion of relational thinking based in an understanding of the equal sign and relational properties associated with operations. In our study we could verify that although the new program explicitly includes algebraic thinking, students are still below what should be for the subsequent formal teaching of Algebra.

Even considering these better results than other studies globally speaking, still they confirm that many students possess an interpretation of the equal sign included in an operational view instead of relational (as other quoted studies). If it's the case, our results could also reinforce the conjecture that students' difficulties may be related to the way they were taught and not an expression of any inability to understand relations between quantities (Molina, Castro and Ambrose, 2006; Schliemann, Carraher and Brizuela, 2007; Haylock, 2006; Ma, 2009). According to the results in items (a), (d) and (g), we could say that students are not aware of using the property a+ b = (a+1) + (b-1), and some errors are due to the operational perspective they assumed.

Such work should be linked to the development of algebraic thinking, because as Kieran (1992) points out, the distinction between arithmetic and algebraic thinking is the change from a procedural vision of operations towards a structural one.

These results seem to indicate that there is still a long way to go before we get good student performances. But for that more care is necessary in the teacher's practices to the equal sign, to numeric relations and to the development of student's algebraic thinking.

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