YOUNG PUPILS' GENERALISATION STRATEGIES FOR THE 'HANDSHAKES' PROBLEM

Thérèse Dooley, St. Patrick's College, Dublin

In describing the strategies that young pupils employ for algebraic generalisation, most research focuses on linear rather than quadratic problem situations. The strategies identified by Lannin, Barker and Townsend (2006) for the cube sticker (linear) problem include counting, recursion, chunking and development of an explicit formula. However, erroneous solutions are often given for 'high' numbers because of the tendency by students to use inappropriate proportional reasoning in such instances. In this paper I describe the strategies that pupils aged 9 – 10 years used in whole-class conversation to solve the 'handshakes' problem, an example of a quadratic generalising situation. They used many (but not all) of the strategies that apply to linear problems and were able to verify the explicit formula that they developed with reference to the structural elements of the problem.

INTRODUCTION

The capacity of children to engage in sophisticated generalisation activities is at the core of the inclusion of the strand of algebra in primary mathematics curricula (e.g., Kaput, Blanton, & Moreno, 2008). In this regard, previous discussions at the Algebra working group of the Congress of European Research in Mathematics Education (CERME) have called for students to engage in problems the purpose of which is to express generality (e.g., Puig, Ainley, Arcavi, & Bagni, 2007). In this paper I describe the strategies that pupils aged 9 – 10 years used to solve the well-known 'handshakes' problem. It builds, in particular, on a recommendation made by Barbosa (2011, p.427) at CERME 7 that students be provided with "tasks which allow the application of a diversity of [generalisation] strategies". However central to the development of these strategies are (a) the role of the teacher and (b) task design.

GENERALISATION STRATEGIES

In early years classrooms, the emphasis in algebra is usually on the exploration of simple repeating and growing patterns. Since any variation usually occurs within the pattern itself, the focus is on single variational thinking (for example, finding relationships between y values rather than between x and y values). As pupils move through the primary school system, greater emphasis is placed on the formation of functional relationships and the generalisation of patterns. Functional thinking is described by Smith (2008, p.143) as follows:

> … representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of relationships across instances.

Functional thinking is intrinsic to algebraic reasoning because it allows for the generalisation of a relationship between two varying quantities. Although young children are capable of thinking functionally (Blanton & Kaput, 2004; Warren, 2005), there is evidence that, throughout the primary years, they focus on pattern spotting in one data set rather than on the relationship between an element of a pattern and its position. Warren (2005) suggests that that this is the case either because single variational thinking is cognitively easier for children or is so engrained from school experience that there is a tendency to revert to it.

Most of the research on the generalisation strategies used by pupils is based on linear problems (e.g., Lannin, Barker, & Townsend, 2006b; Rivera & Becker, 2008). An example of one such problem is the Square Matchstick Problem:

Figure 1: 'Square Matchstick' pattern

The number of matchsticks required to form the nth element of this sequence is $3n + 1$ (for example, 13 matchsticks are required to build the fourth element).

Lannin, Barker and Townsend (2006a) developed a framework of generalisation strategies for linear problems as follows:

- Counting: Student draws a picture or constructs a model to represent the situation and then counts the desired attribute.
- Recursive: Student describes a relationship that occurs in the situation between consecutive values of the dependent variable.
- Chunking: Student builds on a recursive pattern by building a unit onto known values of the desired attribute. In the matchsticks problem, if a student knows that ten matchsticks are required for the third element, s/he might calculate the number required for the fifth element by using a strategy such as 10 + 2 (3) (because the number increases by 3 each time).
- Whole-object: Student uses a portion as a unit to construct a larger unit using multiples of the unit. For example, the fifth element requires 16 matchsticks and therefore the tenth element requires 32 – 1 (where the 1 is subtracted to take account of the first matchstick used in the first element).
- Explicit: Student constructs a rule that allows for immediate calculation of any output value.

According to Lannin et al. (2006b), recursive rules involve "recognising and using the change from term-to-term in the dependent variable" (p. 300) while explicit rules use "index-to-term reasoning that relates the independent variable to the dependent variable(s), allowing for the immediate calculation of any output variable" (ibid). For example, in the square matchsticks problem above, an example of the recursive rule is that the difference between the number of matchstick required is 3 ('going up in threes') whereas an explicit rule is '3n + 1'. The 'chunking' and 'whole object strategies' are similar and represent attempts by students to calculate values immediately. However, they are strategies that often lead to erroneous calculations. A student using a chunking strategy might add the fifth output value (16 in the matchsticks problem) to the tenth output value (31) to find the fifteenth output value (thus finding a solution of 47 instead of 46). A false ‘whole-object’ technique that is prevalently used by students is the application of direct proportion or linearity (for example, doubling the number of matchsticks required for ten to find that for 20 and failing to take account of the ‘first’ matchstick in the pattern). Stacey (1989) suggests that this tends to be evoked when ‘far generalisation’ (an input value that renders the step by step approach unfeasible for example, finding the number of matchsticks required in the 50^{th} element) is required. She found this to be the case even when students have made correct use of counting or a functional rule for smaller input values. This inappropriate proportional reasoning (or the ‘illusion of linearity’) has been found to exist among students of different ages and in a variety of mathematical domains (De Bock, Van Dooren, Janssens, & Verschaffel, 2002).

While the strategies outlined here represent increasingly sophisticated means of generating solutions for any *n*, Lannin et al (2006a) recognise the need not only for students to formulate rules but also to engage in explanation and justification of these rules. In this regard, Rowland (1999) makes a distinction between empirical generalisation - which is achieved by considering the form of the results - and structural generalisation which is made by investigation of the underlying meanings, structures or procedures of the problem at hand. For example in the ‘square matchstick’ problem, a student might notice the ‘going up in threes’ pattern (empirical) or be able to verify the pattern in terms of the need to add three sticks in order to make a new square (structural).

There is little research on the kind of generalisation strategies that pupils might use for non-linear patterns. In research on how primary school pupils abstract mathematical entities in the context of teacher-led discussion, I taught a series of lessons in three different primary schools in Ireland (Dooley, 2010). My research design – a teaching experiment – entailed the development of a hypothetical learning trajectory in advance of each lesson (Cobb, 2000). The one that I formed in advance of a lesson on a non-linear problem (the ‘handshakes’ problem) was based on Lannin et al’s framework. I used it in conjunction with RBC (Schwartz, Dreyfus, & Hershkowitz, 2009) for analysis purposes; however, in this paper I report only on the suitability of the framework as a learning trajectory in this particular situation.

BACKGROUND

The lesson which is the subject of this paper is Chess (the ‘handshakes’ problem). It reads as follows:

> In a chess league each participant plays a game of chess with all other participants. How many games will there be if there are 3 participants? 10 participants? 20? Is there a way to find the number of games for any number of participants?

While the 'Chess' problem is a quadratic problem situation and therefore could be expected to be more cognitively challenging than linear problems for pupils, there are many ways that the problem might be solved. I refer here to those that are most likely to emerge in a primary school setting. This is not to exclude the possibility that a primary student might notice or use others. One way, as shown in table 1, is to make a list and look for a pattern:

Number of People (*x*)	Number of Games (*y*)	Difference (*d*)
1	0	
2	1	1
3	3	2
4	6	3
5	10	4
6	15	5
7	21	6
8	28	7
9	36	8
10	45	9

Table 1: Table of values for 'Chess Problem'

The differences, *d*, between consecutive values *y* form the sequence of natural numbers. A related observation is that a number in the *y* column might be found by adding *x* and *y* values in the previous row (e.g., 7 + 21 = 28, referred to in future as the *x* + *y* method). These patterns are contingent on writing *x* and *y* values in consecutive order. The function mapping *x* to *y* is

$$y = {}^{x(x-1)}\!/_{2}\ .$$

This might emerge from inspection of the relationship between the *x* and *y* values. The same formula emerges if one gives consideration to the symmetric nature of the activity, that is, there is one game for each 'pair' (Rowland, 2003).

Another way to solve this problem is to consider the number of games played by each person, that is, the first person plays a Chess game with seven others, the second with six more, the third with five more and so on. The solution for eight people then is 7 + 6 + 5 + 4 + 3 + 2 + 1 giving a total of 28 games. Although this method (referred to in future as 'summation') generalises for all numbers it becomes cumbersome for larger numbers, especially if the numbers are added in consecutive order.

While the lesson took place over two consecutive days, a related lesson that took place a month earlier with this class was one entitled Friendship Notes. 'Friendship Notes' reads as follows:

> As part of Friendship Week in Greenville School, each pupil writes a short note to each other pupil in his/her class. Each pupil is given one sheet of paper for each note. How many sheets of paper are needed if there are 5 pupils in a class? 10 pupils? What would be the number [of sheets] for any number of pupils?

Both Friendship Notes and Chess are characterised by non-reflexivity (that is, no element of a set relates to itself). The main difference between the activities lies in the property of symmetry. 'Chess' is symmetrical because if A relates to ('competes with') B, then it follows that B relates to A. However, in 'Friendship Notes', if A relates to ('writes to') B, the reciprocal relationship is not implied. For this reason the function mapping x (the number of people) to y (the number of notes) in the Friendships Notes is $y = x(x\text{-}1)$ while in Chess, $y = ½\ x(x\text{-}1)$ where x represents the number of people and y the number of games. Of relevance to this paper is that pupils had developed a 'rule' – expressed verbally – for the solution of any x in Friendship Notes.

As previously mentioned the Chess lesson took place over two consecutive days (the daily sessions will be referred to hereafter as Chess 1 and Chess 2). The format of both Chess 1 and Chess 2 was introductory whole-class discussion and small group work followed by whole-class discussion. In group work pupils worked in self-selecting pairs or triads. In Chess 1, there was a whole class lesson in which consideration was given to the number of games that would apply to 1 – 5 players. Pupils than worked in groups to consider cases of 1 – 10 players – for this part of the lesson they filled in a table similar to that shown in Table 1 above (without the 'difference' column). At the end of the lesson there was a plenary discussion on the number for 20 players. In Chess 2, there was a review of the previous day's lesson. During the group-work phase they were asked to fill in a table for 11 – 20 players and during the final discussion the focus was on the generation of a rule for *any* number of players. There were 31 pupils in the class and the school was located in an area of middle socio-economic status[1]. Data collected included audiotapes of whole-class and small-group conversations, pupils written artefacts, field notes and digital photographs of activities. Video data were not collected due to ethical constraints.

THE LESSON

There was a total of 712 turns in the whole-class discussion phases of both lessons. Up to turn 196, the count strategy was employed. For example, when considering the number of games for four competitors, Killian erroneously suggested five games but used a counting procedure[2]:

136	Killian:	Five.
137	TD:	Why do you think five?
138	Killian:	Cos Enda plays three and then Barry plays two, then David plays Colin , that's ( )
139	TD:	So just say that, explain that to me again ... you've got …?
140	Killian:	So David plays the other three.

141 TD: So how many games is that?
142 Killian: Three.
143 TD: Right, go on.
144 Killian: And then Barry plays the other two and then Colin plays Barry…six.

In turn 196, Anne used an incorrect recursive strategy and suggested that nine games would be played by four competitors:

196 Anne: Em, nine.
197 TD: Why are you thinking nine?
198 Anne: Because it's going up in threes.

The first time that there was any allusion to an explicit rule was in turn 206 when Fiona used the summation rule. She first counted the five pupils at the top of the room but then stated a way of working in which no mention of pupils was made:

202 Fiona: Well, if Enda would play four people and then Barry would have to play eh … three people …
203 TD: Yeah.
204 Fiona: … and then Colin would have to play two people and then David would have to play one.
205 TD: So what do you think it would be? How would you find out the answer? ... What would you do to find out the answer?
206 Fiona: Eh, four, three, two and one.

During group work, most pupils used the $x + y$ recursion method to find solutions for seven, eight, nine and ten competitors. However, when having a whole-class discussion at the end of the first session about a larger number of competitors (i.e., 20), there was evidence of inappropriate linear reasoning on the part of some pupils. For example, Desmond doubled the number of games for ten competitors to find the solution for 20:

357 Desmond: Ninety.
358 TD: Getting ninety, why do you think it's ninety for twenty people?
359 Desmond: Eh ten is forty-five.

The explicit rule that was being expressed towards the conclusion of the lesson related to 'summation'. For example, Fiona used the formula she had developed for five competitors to find the solution for 20 competitors:

402 Fiona: Well you could em you could do em add one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen up to nineteen.

Brenda, Myles and David articulated a similar strategy. Anne suggested multiplying 20 by 19 to find the solution and a short while later, David tentatively suggested a formula:

432 David: Em, well, it might not work all the time but twenty nineteens is three hundred and eighty and half that is one hundred and ninety …

However, he did not justify his conjecture structurally.

In Chess 2, the algebraic strategies most often used were ‘recursion’ and ‘explicit rule’. The counting strategy was not evident. In the lesson introduction, Liam used the $x + y$ method recursively:

447 Liam: Just really one plus zero is one, two plus one is three, three plus three is six, four plus six is ten, five plus ten is fifteen, … eh six plus fifteen is twenty-one, eh seven plus twenty-one is twenty-eight, nine eh eight plus twenty-eight is thirty-six, nine plus thirty-six is forty-five.

During group work, pupils were given a worksheet in which they were asked to give consideration to the number of games of chess that would be played in the case of 11 to 20 competitors and thereafter to 40 competitors. The majority of them used the $x + y$ strategy to complete the worksheets for 11 to 20 competitors. However, in the follow-up discussion on the number of games for large numbers of competitors, David reiterated the rule he had found on day 1:

635 David: Multiply it by the number less …

636 TD: Hm, hm.

637 David: … and then half it.

Shortly after this, Enda observed the similarity between Chess and Friendship Notes.

639 Enda: It looks like … it’s pretty much the very same as the friendship cards, it seems kind of like that.

Barry reflected on the non-reflective nature of both activities, that is

641 Barry: It’s kind of the same thing as, eh, you wouldn’t have to do themselves so there’s going to be one less.

Both Myles and Colin gave consideration to the structural difference between the two activities. Colin gave the following description:

651 Colin: Em, well cos in the friendship notes you have to give two because if there were three you would have to give one to each person …

652 TD: Hm, hm.

653 Colin: … and everyone has to give one to each person, so it’s the same as three by two

654 TD: Hm, hm.

655 Colin: Eh, and in chess you only have to play them once even if they challenge you

656 TD: Hm, hm.

On foot of these ‘structural’ deliberations, Enda announced that

662 Enda: Eh well, I actually definitely agree with David’s way by doing the friendship notes, the same way as the friendship notes and halving it …

It would seem that the justification offered by Barry, Myles and Colin was sufficient to convince him that the explicit formula was indeed appropriate. In follow-up reflective accounts most pupils aligned themselves with ‘David’s method’, although it is unclear if they took account of the structural verification of his formula.

DISCUSSION

The strategies that were used by this group of pupils to solve the Chess problem included counting, recursion, whole object (incorrectly) and formation of an explicit rule. They did not appear to use the 'chunking' strategy. The explicit rules included both summation and a more general formula. Towards the conclusion of the lesson some pupils were able to verify the explicit rule (formula) with reference to the structure of the problem. Although most pupils aligned themselves with the formula at the end of the lesson, it is not clear that all had fully understood the structural dimension. What this paper shows, however, is the variety of strategies that pupils can use to solve complex problems. It could be argued that the format of the lesson (that is the use of consecutive lower numbers initially and later 'higher' numbers) lent itself to a progression from count through recursion and eventually to the use of a formula. In this regard, Warren, Cooper, and Lamb (2006) have alluded to the consecutive listing of x, y values as a factor that inhibits the development of relational or functional thinking. What seems to be the case in this lesson was that, in general, pupils' choice of strategy was based on that which seemed to be most efficient for the task at hand. However, the dominance of the $x + y$ method can be attributed to the use of a table and it would be interesting to see what kind of strategies would emerge if the lesson were designed differently.

In the lesson described in this chapter, the pupils used natural language to express the relationship between the number of players and the number of games. For David's statement in turns 635 and 637 ("Multiply it by the number less … and then half it") could be described in conventional algebraic terms as $\frac{1}{2}\, n\, (n\text{-}1)$. While there well might be concern about the lack of rigor in children's use of terms, I would argue that pupils need to have the opportunity to speak thus before embarking on more conventional symbolic terms and that it is perhaps the lack of opportunity to do so that has contributed to the 'Algebra Problem' (Kaput, 2008). This resonates with an argument, made by Caspi and Sfard at CERME 7, that pupils' informal discourse can serve as a powerful resource for the development of more formal algebraic ideas (Caspi & Sfard, 2011).

Lannin (2005) found that students rarely justify their generalisations in small group situations and acknowledged the role played by a teacher in pressing for such justification. It was similar in the lesson described here and, in fact, in the plenary phase at the end of each session, probes by the teacher led to students justifying their choices and building on each other's thinking (e.g., in turn 641 Barry built on Enda's conjecture). This was a significant factor in developing algebraic thinking. Furthermore, the use of an activity with some variation a month previously laid the ground for structural verification of the formula. It is likely that were it not for the Friendship Notes lesson, pupils' thinking would not have progressed beyond the empirical level. To this end, the sequencing of activities by teachers is an important element in stimulating and developing pupils' generalisation strategies.

1: In the Republic of Ireland, indicators such as unemployment levels, housing, number of medical card holders and information on basic literacy and numeracy are used to determine socio-economic status of schools.

2: Transcript conventions (related to this paper) are: TD: the researcher/teacher (myself); … : a short pause; ( ): inaudible input.

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