ABOUT STUDENTS' INDIVIDUAL CONCEPTS OF NEGATIVE INTEGER – IN TERMS OF THE ORDER RELATION

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In the presented study we investigated sixth graders' individual concepts of negative integer right before they were introduced to the "world" of negative integers. In order to investigate students' first ideas of negative integers, we initially investigated their ideas concerning the order relation of integers. With a qualitative data analysis utilizing a theoretical lens concerning individual concept formation, we gained insight into the students' individual procedures and conceptions as well as into how the procedures are linked to the students' previous knowledge.

INTRODUCTION

In mathematics education there are many studies dealing with the development of specific content areas (i.e. algebra, fractions and functions) and other studies analyzing learners' perspectives. Many of these empirical works focus on the outcomes of students' learning, but not on the learning *processes* themselves. Furthermore the misconceptions and difficulties in individual learning processes are analyzed with respect to an ideal learning trajectory, whereas the productive potential of these individual processes is not taken into account. In our contribution the aim is to structure and understand students' learning processes in mathematics. Thereby we use a theoretical approach, which enables us to reveal key points of misunderstandings as well as main influencing factors. These constitute the basis for an adaptive structuring of the mathematical content and hence for the development of instructional designs.

NEGATIVE INTEGERS AND THE RESEARCH OBJECTIVE

Negative integers constitute a significant topic in mathematics teaching and learning. The concept of negative integer is relevant for both, the handling of inner mathematical situations such as solving the equation x+3=1, as well as the handling of real-world situations, e.g. comparing temperatures below zero or situations with debts and assets. Therefore, the *introduction* of negative integers has often been suggested not to beg big difficulties for students, as Fraenkel (1955, 68) notes: „Negative integers are nowadays considered as a rather simple subject which may be taught even in elementary schools. It apparently contains no difficulties, except for the multiplication of a negative number by a negative." But several studies show low success rates of students solving computations involving negative numbers (Human & Murray 1987, Murray 1985), which indicate „deeply-rooted and widely-held misconceptions" (ibid., 168). These difficulties are not restricted to the above-mentioned multiplication – which *does* indeed constitute a big challenge for students – but they also concern other contents like the addition and subtraction of integers. But these findings about teaching and learning of negative numbers are rare (Bruno &

Martinón 1996, 98) and still not sufficiently clarified (Kishimoto 2005, 317). In terms of the teaching and learning of integers, there can be spotted (amongst others) two main *perspectives*, which constitute the starting points of the present research project.

The first perspective concerns students' individual ways of proceeding and their difficulties as well as possible *causes* for problems. While there are several studies examining the students' achievement in solving tasks by analyzing the solving rates and considering students' performances, strategies and difficulties in solving equations (e.g. Borba 1995, Murray 1985), there is *still* a deficit in qualitative research concerning the sophisticated interplay of concepts that students use in order to handle situations involving integers. The attempt to get such an insight is born of the constructivist view that the learning process is an individual, but also social affected construction process and that every student is supposed to construct his or her specific, individual concepts in order to structure the world (of mathematics) on her or his own.

The second perspective concerns the very beginning with negative numbers in mathematics class. It is our special interest to investigate the students' levels of knowledge when being introduced to negative integers in school and to analyze which parts of their previous knowledge students take into account (cf. Bruno & Martinón 1996). Likewise, we focus on students' understanding related to the *order relation* of integers, which constitutes – followed by addition, subtraction etc. – one of the first topics when learning about the negatives. The suitable continuous ordering (in terms of the formula $a<b\ (a,b \in \mathrm{IN}) \Rightarrow -b<-a$) is not self-explanatory for students, who may order the numbers as well by focusing on magnitudes (in terms of the formula $a<b\ (a,b \in \mathrm{IN}) \Rightarrow -a<-b$). "Ordering negative numbers is complex because there are two possible orderings that are supported by thinking about common contexts – the standard ordering and ordering by magnitude (absolute value)" (Widjaja, Stacey & Steinle 2011, 81). While there are already some findings about the ordering of negative integers (see Bruno & Cabrera 2005, Thomaidis & Czanakis 2007, Widjaja et al. 2011), an access into the *interplay of concepts* is still missing: In order to learn more about the very beginning with negative integers, there has to be gained more insight into the parts of the students' *previous knowledge* taken into account while ordering integers, and to the reasons *why* students use the one or the other ordering.

These two perspectives – the focus on the procedures as well as the focus on the students' levels of knowledge before being introduced to negative integers in school – constitute the basis for our investigation. The expected findings are assumed to serve for a structuring and organization of the mathematical content of the introduction of negative integers, especially of ordering integers.

The research issues of the present study are the following.

(1) Which *procedures* do students have for ordering integers? Which concepts and which assumptions are interwoven in these strategies? Which difficulties do students have?

(2) Which *previous knowledge* is taken into account? Which reasons can be reconstructed for the students' approaches to order integers? Which previous-built concepts are taken into account?

In our research project we pursue these questions with regard to two main interests: Firstly we want to figure out, how students handle the situations just shortly *before* they get to know negative integers in school in order to bring their previous knowledge to light. We intend to assess how students perform due to this previous knowledge without having had any introduction to the topic of negative integers before. Secondly we intend to focus on *concept formation processes*. It is our aim to investigate these processes on the one hand in a short-term perspective – to analyze how difficulties and misconceptions can be overcome and which impetus is helpful. On the other hand we focus on a long-term development over a lecture series and intend to find out how procedures and utilized concepts change. Because of the limited space, we have to restrict ourselves to the findings in terms of the procedures and the activated knowledge *before* an introduction into the "world" of negative numbers in the presented paper.

METHOD

For the purpose of our investigation, half-structured, task-orientated clinical interviews (cf. Selter & Spiegel 1997) were undertaken with eight sixth graders in German schools. The interviews were followed by a lecture series to introduce negative integers by means of a suitable learning environment, which again was followed by interviews. The qualitative analysis of video sequences and transcripts was taken as a basis for evaluating the data.

For the purpose of this study, students got different pairs of integers and had to determine the "greater" one. For these tasks, the kind of *representation* of the displayed numbers played an essential role. As "non-positive integers are not representable concretely as manipulable objects" (Davidson 1987, 431) and are in a way 'fictive' as they are not physically perceptible on their own, representations have a pivotal importance for the teaching and learning of negative integers. Hence "it is important that schematic visualizations … take the place of missing empirical objects" (Malle 1988, 302). There are most notably four kinds of representation, which seem to be meaningful for getting to know the negatives (expanded from Bruno 1997): These are (a) the representation on the number line or other ordinal arrangements regarding the order of integers (like ... -3 -2 -1 0 1 ...) (b) a quantity representation, which students mostly know from natural numbers (like e.g. "3 means three spots"), (b) the representation in a real-world context (e.g. temperatures, debts-and-assets) as well as (c) the symbolic representation (e.g. -6 or "minus six"). These kinds of representation and their interplay are of great importance for the learning of negative integers (cf. Bruno & Martinón 1999).

Within or aim to investigate the students' ways of proceeding and activating their previous knowledge before the introduction of negative numbers, we particularly wanted to get information about whether students were already able to interpret a symbolic representation of the negatives. Whereas Malle (1988) found that all of the interviewed students in his study knew the symbolic representation of negative integers before having had an introduction to negative integers, the aim of our investigation was to find out, if the students in our study did so as well. In our investigation we also intended to find out, if students are able to utilize the number line or other ordinal arrangements, and in what way they fall back on real-world contexts. For this purpose, the integers were presented in a symbolic way and the use and change of representations was investigated.

Naturally, by displaying integers in a symbolic way, the knowledge students are able to activate, is limited: If integers were for example given in a contextual way, students would surely be able to handle situations more competently, using real-life knowledge more easily. But in our study we wanted to figure out, which knowledge students are able to activate when regarding numbers such as 12 or -15 [1]: if they alter the kind of representation for their reasoning, if they activate contextual knowledge etc.

In the interviews students first compared a positive with a negative integer, then two negatives, followed by the comparison of a negative and a positive integer each with zero. In doing so, students received two small cards each of them displaying an integer in a symbolic representation (e.g. 12 and -15).

A GLIMPSE OF THE ANALYTICAL FRAMEWORK

For the concern to analyze students' individual concepts and their development we use a theoretical approach (Hußmann & Schacht 2009, Schacht 2012), which serves as a theoretical lens for analyzing concepts and from which appropriate analytical tools can be deduced. The theoretical approach is based on different influences: For the epistemological framework, Robert Brandom's semantic Inferentialism (e.g. Brandom 1994, Sellars 1999) constitutes a basis and an essential factor from a philosophical perspective. Gérard Vergnauds Theory of Conceptual Fields (e.g. Vergnaud 1996, 1997) replenishes the theoretical framework from a rather learning theoretical and psychogenetic point of view. Because of space constraints, the theoretical framework, which is supported by philosophical and psychological considerations, cannot be presented here (cf. Schacht 2012). It can only be outlined, that its kernel signifies that understanding a concept means understanding the use of a concept with its reasons and its inferences.

In the following, the main building blocks of our analytical scheme are outlined by giving a glimpse of the analysis of one student's way of proceeding during the preliminary interview. The case of the student Nicole, who is a low achiever in mathematics, was chosen to be presented here. While working on the task to name the greater of a positive and a negative integer in the preliminary interview she proceeds in a way, which is interesting as she seems to subtract the two integers and compare them by using the difference.

Interviewer: I brought along two numbers and I want you to tell me which one of them is greater (gives two small cards to the student, displaying 12 and -15)

Nicole: (regarding the cards) (2sec) The fifteen. (6sec) No the twelve, because there at- in front of the fifteen there is- minus fifteen is displayed there- then it has to be the twelve.

Our approach is based on the assumption that concept formation takes place within situational conditions and that a strong interrelationship exists between situated action and conceptualization. To understand students' concept formation it is indispensible to investigate in which situations it takes place and on which aspects of the situations students focus. By setting focuses in situations, students on the one hand deal with the situation at hand and on the other hand they simultaneously deal with the concepts they possess. While trying to handle available situations, students use individual concepts, relationships, properties etc. as categories that they developed before and that enable them to select the diverse information embodied in the given situation. ***Focuses*** are categories (properties, concepts, relationships, etc.), which are used to handle and select the information of a given situation (e.g. 'the minus sign' or 'subtraction' or 'the number line' or 'changes in temperatures'). In the given interview sequence Nicole seems to focus on the ***minus sign***.

Besides individual focuses the student makes assumptions concerning the situation, about relationships of the properties etc. They are interwoven with the focuses since focuses give the orientation for the assumptions to be made. In our approach, we are looking at these assumptions by examining individual ***commitments***. ***Commitments*** are propositions that are held to be true – they are individual judgments, which are made explicit by the student (e.g. "Negative numbers are below zero." or "Zero is the freezing point."). In the above-mentioned case of Nicole, there can be assigned the following commitments to her utterances: Firstly, she seems to commit to *"Among (-)15 and 12, (-)15 is greater." (cm01)* At this point in time, it is not yet reconstructable whether she means that 15 or minus 15 is greater. But she seems to reject this commitment by uttering "no" and committing to another, contrary commitment.

Then she commits to *"Among (-)15 and 12, 12 is greater."(cm02)* and obviously tries to entitle this commitment by giving a reason for it. In other words, the latter commitment *(cm02)* constitutes a conclusion and she is trying to give a premise for it. For our theoretical approach these entitlings have a pivotal importance as they concern the reasons that the students give for their commitments and focuses and finally for their approaches. There is an ***inferential relationship*** between commitments if one commitment entitles the student to affirm another one (e.g.

“Negative numbers are below zero *because* positive numbers are above zero.“) and if the student accepts this inference as true. Practical reasoning discloses such inferential relationships.

In Nicole’s case, she seems to entitle the commitment “*Among (-)15 and 12, 12 is greater.”(cm02)* by committing to the fact that there is a minus (sign) in front of 15. The commitment *“There is a minus sign in front of 15.”(cm03)* is assigned to her utterance. The inferential relationship between the commitments is assigned as an inference: *“Among (-)15 and 12, 12 is greater, <u>because</u> there is a minus in front of 15.”*

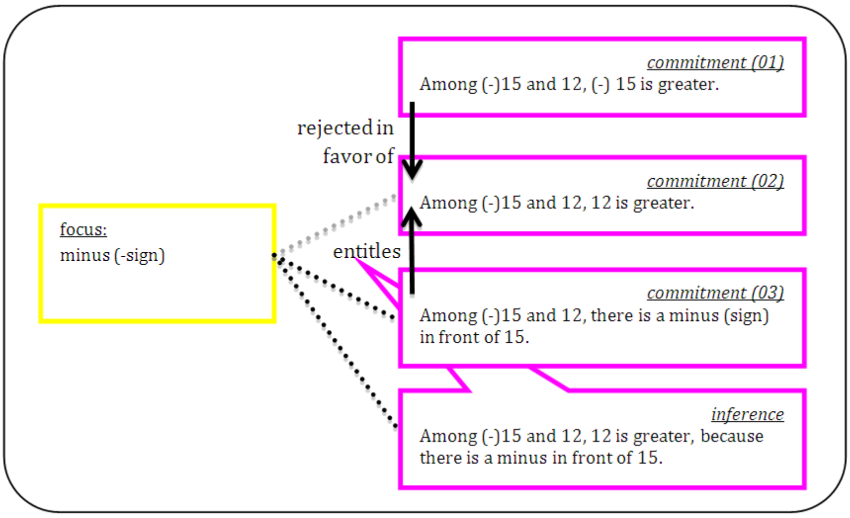

Figure 1 Nicole’s focus, commitments and inference

Focuses, commitments and inferences form inferential webs. Understanding the concept of integer means to be endued with an ***inferential web*** in which the concept of integer itself and other related concepts are involved. [2] By analyzing the students’ individual inferential webs it is possible to get an insight into the above-mentioned research issues: It enables us to uncover the students’ individual approaches in detail as well as their difficulties (research issue 1) and they make it possible to focus on the previous knowledge that the individual approaches rely on (research issue 2).

The outcomes of the analysis concern the above-mentioned research perspectives as they give a detailed insight into individual procedures, into difficulties and possible reasons especially for the point in time when getting to know negative integers. By examining inferential webs, it can be investigated which previous concepts are taken into account and how the inferential structures change during the learning process. This constitutes the basis for structuring the mathematical content as well as for the optimization of the developed learning environment.

Nicole’s strategy of subtraction

At a later time, the analysis unfolds *why* the focus on the minus sign leads Nicole to the inference *“Among 12 and -15, 12 is greater because there is a minus in front of*

15" (see above). The commitments "*In order to determine the greater number among -15 and 12, I can subtract."* with the focus on *subtraction* as well as "*The numbers 12 and -15 form a mathematical task."* with the focus *mathematical task* that can be assigned to her utterances seem to have a pivotal importance. It appears that she perceives the situation as a mathematical task, which prompts her to subtract the two numbers that she had laid down side by side, the 12 left. It also becomes clear *why* she seems to focus this way: There are three commitments and corresponding focuses which point out why Nicole is thinking about a mathematical task and – as a consequence – subtracts. These are (a) "12 is a number", (b) "I have got the minus of the mathematical task from the -15", (c) "I subtract 15". While regarding the numbers 12 and -15, she pays attention to what she already knows: She first focuses on the 'normal' *number* 12 as the first part of a mathematical task. Then she focuses on the *minus sign*, which is required for a mathematical task. For her, the 'minus' is linked to the focusing of *subtraction* – it indicates a subtraction in terms of a binary use of the minus sign (Bofferding 2010) as an operative sign (Vlassis 2004) and *not* to the focusing of a signed number, where the minus-sign is a predicative and unary one (ibid.). By means of the further analysis of Nicole's inferential web, her approach concerning the comparison of the two integers as well as the influence of her previous knowledge can be increasingly detailed. In case of a negative and a positive integer the inferential web likewise shows that she uses a conspicuous strategy of first subtracting, then replacing the minuend with the result of the task und comparing the re-placed minuend (the result) with the subtrahend subsequently. The inferential webs of other classes of situations (e.g. comparing two negative integers) show, that she uses partly the same focuses but adapted commitments according to the situation involved. The case of Nicole shows that she seems not yet to know the minus sign as a predictive sign. Because of that she naturally is not able to interpret the integer as negative integer, but she proceeds in a comprehensible way, which makes sense from her individual perspective.

Tom and his recourse to a real-life context

In order to give a glimpse of the broad range of the students' previous knowledge, Tom's way of comparing the numbers 12 and -15 is mentioned in the following.

Interviewer:	I brought along some cards, and on the cards there are displayed two numbers. And I want you to tell me which one of them is greater (gives two small cards to the student, displaying 12 and -15)
Tom:	(receiving the cards) May I -
Interviewer:	Mhm (affirmative), you can take them.
Tom:	(regarding the cards, holding them in his hands) The twelve is greater.
Interviewer:	Mhm? (quietly)
Tom:	Because (slowly) minus is below zero and the twelve is just normal, the twelve.

Interviewer:	Okay. How do you imagine that?
Tom:	Well at the thermometer you sometimes also see minus numbers and then, if there, er, is no minus there, then there is simply just the number. Twelve degrees or so.

When regarding the cards briefly, Tom at once seems to commit to "*Among -15 and 12, 12 is greater".* In order to justify this commitment, he gives two reasons, which he combines. The inference "*Among -15 and 12, 12 is greater* <u>*because*</u> *minus numbers are below zero* <u>*and*</u> *12 is a normal number."* can be assigned to his utterances. Asked how he imagines that, he mentions the real-life situation of negative numbers on the ***thermometer***. Commitments like "At the thermometer there are minus numbers." and many more can be assigned to his utterances in this excerpt of the transcript and the following ones. For him, the real-life context of temperatures, especially the *thermometer*, the concept of *natural number* ('normal number') and the concept of *negative number* ('minus number') seem to constitute important focuses, which characterize his way of proceeding when trying to determine the greater of these two integers. Later in the interview it transpires that the real-life context of ***temperatures*** seems to play an important role for Tom in a way that it seems to serve as a fruitful foundation stone for Tom's inferential web including negatives. Tom's experiences concerning rises and falls in temperature seem to serve as basis for his concept of integer.

THE OUTCOMES OF THE STUDY – AN OVERVIEW

The presented findings were deduced from two case studies: the case of Nicole and the case of Tom. The analysis of the students' inferential webs gives insight into the broad spectrum of the students' levels of knowledge before having had an introduction to negative integers. In the following, some of the main findings are outlined and reflected in the context of possible consequences for the mathematical content. They concern students' procedures and difficulties as well as the involved previous knowledge.

Concept of natural number By the analysis of Nicole's preliminary interview we found that there *are* sixth graders whose inferential webs, which are activated when regarding symbolic represented integers like -12, mainly affect natural numbers. The analysis shows that Nicole does not yet seem to know about negative numbers. Consequently, she is not yet able to interpret the minus sign as a predicative sign. Instead, she activates previous knowledge in terms of arithmetic problems, which makes sense from her perspective: She knows "normal" numbers and she knows about the minus sign as indicator of subtraction which leads her to subtract the two numbers and compare them consequently. She develops a subtraction scheme with its own rules, and she tries to activate and adapt it to new situations. She seems not to know that there are negative integers beneath positive ones. Our findings seem to support the suggestion, that some of the students have to overcome deeply-rooted (mis-)conceptions from elementary school, e.g. that there are no numbers below zero (cf. Bruno 2001, 415). The finding that students partially seem not to be able to interpret the minus sign as predictive sign, indicates that an introduction of negative integers in mathematics class should not purely be based on a symbolic representation, but probably should resume experiences in real-life contexts.

Concept of negative integer On the contrary, the case of Tom shows that there *are* students, who already have a substantial and sophisticated inferential web including negative integers at their disposal – even before they received an introduction to negative integers in school. Tom's inferential web seems to be relatively stable and to a great extent tenable from a mathematical point of view. It is interesting to have a look at the previous knowledge Tom is activating: He utilizes contextual knowledge from the context *temperatures* (see above). This shows how powerful real-life experiences may be for conceptual development.

We also found that the assigned inferential web in Tom's preliminary interview indicates an order relation, which focuses on magnitudes in terms of a "divided number line" model (Vlassis 2004). This indicates that a continuous order relation for integers can not be deduced easily form real-life experiences by students on their own, but it needs educational attention and support.

Because of space constraints the other findings of our study cannot be mentioned here. They can be seen in Schindler (2013).

NOTES

1. In Germany it is common practice to display non-positive integers with a prefixed minus sign such as -8.

2. With 'related concepts" we mean concepts that belong to the conceptual field, in which the concept is involved (see the Theory of Conceptual Fields by Vergnaud 1996, 1997).

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