FOCUSSING STRUCTURAL RELATIONS IN THE BAR BOARD – A DESIGN RESEARCH STUDY FOR FOSTERING ALL STUDENTS' CONCEPTUAL UNDERSTANDING OF FRACTIONS

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For fostering conceptual understanding of fractions, connecting different representations is an often used design principle. This paper shows that this design principle is necessary but not sufficient and should be complemented by focussing structural relations. In a design research study in grade 6, different strategies for this principle were developed and empirically investigated with respect to the generated learning processes. The activities were organized around the so-called fraction bar board, a visual model that allows a comprehensive structural view on fractions, order and equivalence.

Key words: Fractions, structural relations in pictures, design research study

Students' difficulties with conceptual understanding of fractions have often been shown empirically (e.g., Hasemann, 1981; Behr et al, 1992; Aksu, 1997). As a consequence, fostering students' conceptual understanding for fractions has become a central aim for many curriculum design projects all over the world (e.g., Bokhove, et al., 1996; Cramer et al., 2009; Streefland, 1991; Prediger et al., 2013). Before developing algorithmic skills, students should develop meanings for fractions and for the basic fraction concepts like order and equivalence of fractions. These curriculum projects vary in the chosen contexts and in their priorities for different fraction models: Most emphasis is usually given to the part-whole model, whereas varying priority is attributed to other important models like measure, ratio, operator, or quotient in situations of equal sharing (see Behr et al., 1992, for an overview on models). Despite of different choices of prioritized models and of contexts, nearly all curriculum projects share the design principle of *relating multiple representations,* namely graphical, symbolic, verbal and enactive representations (Lesh, 1979). Most of them also refer to the principles of *including students' everyday experiences* by means of suitable contexts and *initiating mathematical discussions* on mathematically *rich open problems* (Freudenthal, 1983).

This paper intends to show that connecting different representations is a necessary but yet not sufficient condition for developing conceptual understanding, because especially weaker students tend not to construct the mathematically intended structural relations automatically. That is why the design principle *explicitly focussing structural relations* should complement the set of design principles. The empirical findings that support the importance of this principle and design strategies for implementing it for the topic 'order and equivalence of fractions' are drawn from several design experiments that were iteratively conducted within the long term design research project KOSIMA (Hußmann, Leuders, Barzel, & Prediger, 2011).

1. THEORETICAL AND EMPIRICAL STARTING POINTS

1.1 Fraction bars as important visual model

As developing conceptual understanding of mathematical objects necessitates to *relate different representations* (Lesh 1979), the concrete choice of contexts and concrete graphical representations is crucial for the mental models that students can develop. In line with many other curriculum projects (see above), our design approach starts with identifying and interpreting fractions in a variety of situations of equal sharing and part-whole situations, being graphically represented in rectangles, circles, bars and other pictures. In the second unit on order and equivalence of fractions (Prediger et al., 2013), the learning arrangement focuses on *fraction bars as the central visual model* that has to be connected to symbolic and verbal representations (bars are also used by other curriculum projects, e.g. van de Walle & Thompson, 1984; Cramer et al., 2009; Bokhove et al., 1996).

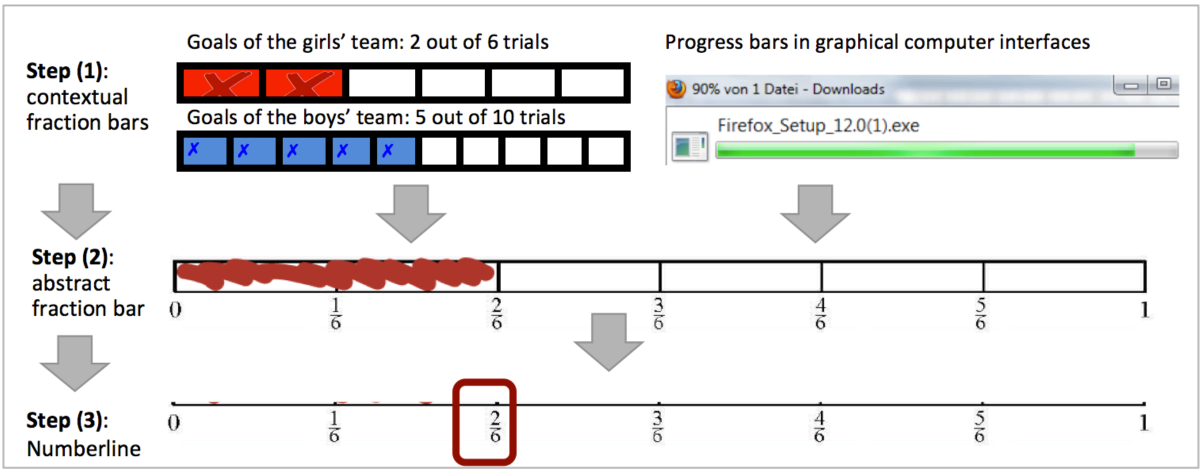


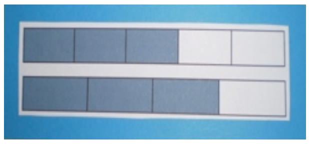
Figure 1. Three steps from contextual fraction bars to abstract fraction bars to the number line

In our curriculum, two everyday context situations support the introduction of fraction bars (Figure 1): (1a) Comparing goals for an unequal number of trials, a *situation* which provokes students to invent the mathematical concept of relative frequencies for measuring fairly, (1b) progress bars of graphical computer interfaces, showing e.g. the progress of a download. (2) From these context situations, we derive an *abstract* fraction bar as the central representation for order and equivalence. (3) In the last step (not treated in this paper), we abstract the fraction bar to the number line and use it as a bridging tool between part –whole and measure model (cf. Keijzer & Terwel, 2003, p. 288). These bridging functions are the main reason for choosing bars as main visual model.

1.2 Limits of graphical representations – a snapshot from an initial case study

Interpreting fraction bars is not from the beginning evident for all students, as could be shown in a case study on low-achieving students (Prediger, 2013). The two boys Cavit and Ismet from grade 7 followed a typical German curriculum with a focus on relating representations, but without real focus on meaning. In the conducted interview, both boys easily assigned the symbolic representations 3/4 and 3/5 to given fraction bars. In the transcript, Cavit explains how to find 3/5 in the bar:

9	Interviewer	... what in this picture *(hints to the fraction bars)* shows you the fraction?
10	Cavit	Ehm...Ehm the ... first, you have to count the small pieces. These are five. And then, the three coloured ones, three fifth.



In spite of this correct answer, Ismet and Cavit compare (in the next sequence) the symbolic fractions 3/5 and 3/4 by commonly deciding that 3/5 was bigger. The interviewer asked them to use the graphical representation to validate their judgement:

17	Interviewer	... how can you see in the picture that this fraction (*hints to 3/5*), when you say, it is greater than this *(hints to 3/4)* How can you see that in the picture?
...		
24	Ismet	[...] because here *(hints to the 3/5 bar)* it is five, and here are four *(hints to the 3/4 bar)* then you see that this *(the 3/5 bar)* is large and this *(the 3/4 bar)* is small [...]

Although line 10 showed the boys' capability to switch between the graphical and the symbolic representations without mistakes, they order the fractions idiosyncratically. Ismet's explanation in line 24 shows that relating representations does not guarantee to see the mathematically intended structural relations in a graphical representation: Instead of comparing the length of the coloured part, Ismet counts the pieces and argues that 3/5 is larger because its whole is divided into more pieces than the one for 3/4. This limit of graphical representations has also been found for other arithmetical topics (Steinbring, 2005): Teaching students to draw correctly is not sufficient to guarantee that they mentally construct the intended structural relations which mathematicians see in a specific graphical or symbolic representation. Cavit and Ismet could identify the right fraction bar and described the drawing procedure. However, their linguistic expression of the relation between the part 3 and the whole 5 was restricted to the word "and" (line 10). Ismet's idiosyncratic interpretation of the *order* and this vague expression are indicators for the structural relation between part and whole not being completely mentally constructed. Hence, further support is needed (cf. Prediger 2013).

The short case study illustrates the necessity to understand 'fraction' as a relational concept (Steinbring, 2005), in the sense that it grasps the structural relation between the part and the whole. Although many students mentally construct this relation simply from dealing with graphical representations, this can be challenging for low-achieving students (similarly Moseley, 2005). From these empirical and epistemological starting points, the following design challenge and empirical research question was concluded for a design research study aiming at overcoming the limits of graphical representations:

How can we foster all students' mental constructions of the intended structural relations between part and whole by initiating activities with fraction bars?

2. METHODOLOGY OF DIDACTICAL DESIGN RESEARCH

Mathematics education research is sometimes dichotomised by *two different aims*: 1. *designing* concrete (teaching-)learning arrangements for mathematics classrooms, and 2. *understanding* and explaining teaching-learning processes. More and more researchers aim to overcome this unfruitful dichotomy and to combine empirical research and the design of learning arrangements in order to advance *both*: practical designs and theory development (e.g. van den Akker et al., 2006).

In our research group, we follow the programme of Didactical Design Research as formulated by Gravemeijer and Cobb (2006) which combines the concrete design of learning arrangements with fundamental research on the initiated learning processes. By *iterative cycles of (re-)design, design experiment and analysis of learning processes,* it focuses on both: 1. creating prototypes of learning arrangements and their underlying theoretical guidelines (design principles and strategies), and 2. elaborating on an empirically grounded *subject-specific local instruction theory* that specifies the epistemological structure of the particular learning content, students' learning pathways, typical obstacles in these pathways, and conjectured conditions and effects of specific elements of the design (Prediger & Schnell, 2013; Gravemeijer & Cobb, 2006, p. 21). For investigating processes initiated within the designed learning arrangements, design experiments have proven to be a fruitful method of *data collection* (cf. Komorek & Duit, 2004; Gravemeijer & Cobb, 2006). We usually start by laboratory settings with 2-4 students as this allows in-depth insights into individual, context-specific learning pathways, obstacles or individual prerequisites (cf. Komorek & Duit, 2004). Once the arrangements have proven suitable to initiate the intended learning processes, the experiments are widened to classroom settings with regular teachers and normal resources for investigating their robustness under varying pedagogical conditions.

For *data gathering* in the topic 'order and equivalence of fractions', we successively conducted 31 design experiment series in laboratory settings (in sum n = 69 students), each with 2-6 sessions. Additionally, long-term classroom experiments were conducted with six classes (n = 123 students) and their regular teachers, encompassing about 36 sessions for the whole fraction curriculum (on basic concepts, order and equivalence for 5 sessions, then addition, multiplication etc.) All design experiments in laboratory settings were videotaped. The data corpus includes the videos, transcripts of selected video-sequences, teaching materials and students' products. The *data analysis* of the complex process data requires interpretative qualitative methods that are specified according to the research interest in each phase of the process (Prediger & Schnell, 2013).

Due to space limitations, the complex, iterative design research process cannot be reported here. Instead, some selected snapshots are presented that are chosen to illustrate three design strategies for implementing the design principle 'focussing structural relations'. Each of the three strategies is illustrated by one selected activity and typical moments in the learning process. These snapshots intend to contribute to a local instruction theory for fostering all students' mental constructions of structural relations for fractions, order and equivalence.

3. DESIGN STRATEGIES FOR FOCUSSING STRUCTURAL RELATIONS

3.1 Constructing relevant structural relations by contexts and systematic variation

The case of Ismet and Cavit suggests that comparing fractions might be an important activity for constructing relevant structures in the part-whole model. The *context* of computer interface progress bars (see Figure 1) was introduced into the material after analysing these initial case. In later design experiment cycles, this connection between progress bars and the symbolic and verbal representations proved to be useful for students constructing the intended structural relations, because students activate everyday experiences and argue for example, “no, 3/5 cannot be bigger, it has less downloaded”.

In addition to these contextual supports, we drew back on the design strategy of *systematic variation* emphasized by Duval for relating representations structurally: “It is only by investigating representation variations in the source register and representation variations in a target register, that students can at the same time realize what is mathematically relevant in a representation, achieve its conversion in another register and dissociate the represented object from the content of these representations.” (Duval 2006, p. 125).

One example for a systematic variation activity is printed in Fig. 2. It was designed to support especially low-achieving students to construct the quasi-cardinal relation between the systematically varied fifths: 1/5, 2/5, 3/5, 4/5 and 5/5 (which is one aspect of measuring). Although this relation is immediately clear for some students, our design experiments have shown that the task can allow an interesting discovery for lower achievers and help to make clear the difference between a divided whole and its parts (Prediger & Wessel, 2013).

More and more fifths

a) Now Kenan produces fifths with fraction bars. Complete his table.
Fraction that Kenan wants to draw: Picture:

$\frac{1}{5}$

$\frac{2}{5}$

$\frac{3}{5}$

$\frac{4}{5}$

$\frac{5}{5}$

b) Examine the table precisely and consider the following: What happens with the coloured part of the fraction stripe? Why does the coloured part change?

c) Your research: How and why does the fraction change? Write down your findings so that another student can understand what is happening with it and why it changes. …

Figure 2. Elementary task for weaker students – example for the design strategy systematic variation

For example, Hadar (12 years old) writes, “When the numerator gets bigger, one gets more fraction.”. Asim (12 years old) explains the differences with reference to a contextual situation: “Because the numerator gets always bigger, that is why Kenan gets always one [piece] more [of the chocolate bar]. And the denominators stay the same.” (citations from a case study in Prediger & Wessel, 2013).

3.2 Embedding structural relations in a comprehensive visual model: Bar board

For comparing fractions with respect to order and equivalence, students need to connect different fraction bars, e.g., not only fifths, but also forths, thirds and eighths. Whereas higher achieving students tend to *mentally* construct the bars quite quickly, weaker students profit from a comprehensive visual model that helps them to see many bars at the same time. Inspired by the prototype of the “fraction lift” which connects some bars (Bokhove et al., 1996), we have developed the fraction bar board as a comprehensive visual model for comparing fractions with respect to order and equivalence. Figure 3 sketches how to find fractions that are equivalent to 1/3 by vertically positioning the ruler. The lamination of the bar board guarantees its long time usability.

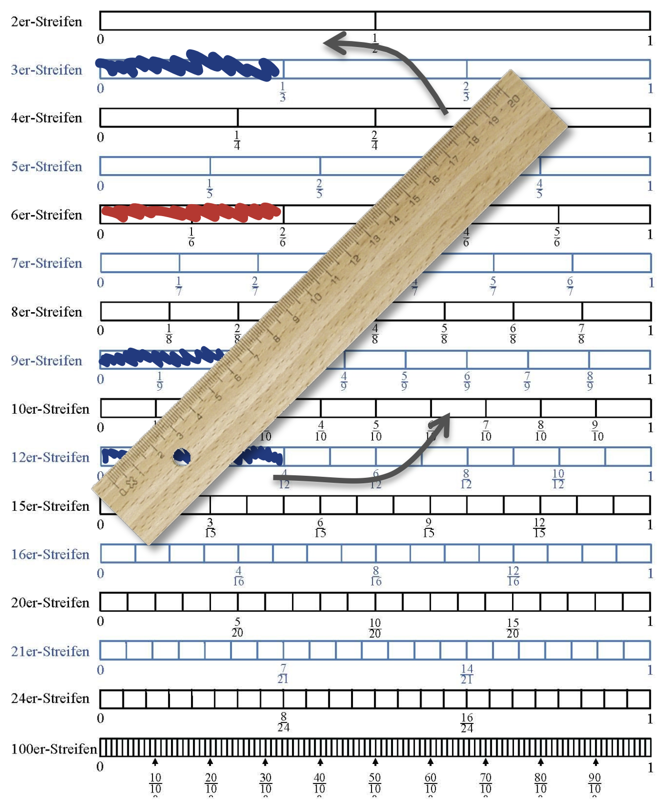


Figure 3. Extract of the fraction bar board

The classroom design experiments have shown that most students quickly learn to use the bar board and understand the ordinal relations of fractions in the interplay of symbolic, verbal and graphical representation (Prediger, 2011). The bar board allows to embed a singular ordinal relation into a comprehensive visual model. By this, students achieve an overall orientation as the example of Lisa illustrates: Having worked with the bar board for two hours, Lisa is asked to compare 1/10 with 2/3. She immediately says without watching the bar board: “Imagining them, it is evident, 1/10 is so much on the left.” (cited from a classroom video).

In the next step of the curriculum, the bar board serves as starting point for the process of progressive schematization (Treffers, 1987) from visually searching equivalent fractions to extending fractions by calculating: Paul explains “If you go from 2/6 to the 12-bar, each sixth transforms into two twelths, thus the denominator must be multiplied by two. The coloured pieces also transform from sixths into twelfths by one into two, so the numerator must also be multiplied by two.” (cited from a classroom video).

3.3 Internalizing structural relations by mental practices

Unlike Lisa and Paul, who quickly internalized the relational structures inherent in the bar board, other students need more help to explicitly focus the structural relations between the bars. This is illustrated by the case of Anna and Jasmin, both 11 years old (in Prediger, 2011). Both girls worked with the bar board, but treated it only empirically, which became apparent when they searched for fractions being equivalent to 3/4:

230	Anna	9/12 here, isn't it? *(marks a sign on the 12-bar and draws a vertical line)* Well, yes, that is... is 6/8, but..... it works, it seems to work! *(controls her bar board with the ruler again, during 12 seconds)*
232		No, it is only wrong by one millimeter.

With purely empirical methods of measuring in the bar board, Anna could not convince herself whether 9/12 is equivalent to 6/8 and 3/4 or not. Whereas other children in this situation started to argue with structural relations (like Paul above), both girls only referred to the bar board as an empirical instrument (Steinbring, 2005). Even later, when they found a rule in the number patterns (multiply numerator and denominator by 2 or by 5) they could not explain the found regularity:

295	A	Yes, denominator and numerator must always be the same. That means here, 3/5 is the double of 6/10. *(8 seconds break)*
...		*(Interviewer asks for an explanation)*
303	A	That's just how it is. Like: why is a banana called a banana *(both girls laugh)*

Thus, Anna's and Jasmin's process of progressive schematization was only partly successful due to a lack of conceptual understanding *why* their rule "take the double for denominator and numerator" applied for all fraction bars. As a consequence of these empirical findings, the redesign searched for a strategy that makes Paul's insight into structural relations more explicit for all children. For this, we developed mental practice activities (Weber, 2011) for internalizing the structure of the bar board (see Figure 4).

The bar board in your head

You can also find equivalent fractions, if you only imagine the bar board in your head. Try it!

a) Imagine how to mark 2/3 on the 3 bar.
Go from the 3 bar to the 6 bar. Where is 2/3 here?
How many pieces does the 6 bar have?
How many of them are coloured?

b) Pose yourself different similar tasks. ...

c) How many 25ths are 3/5?
Explain, how you find the result even if you cannot imagine the bars.

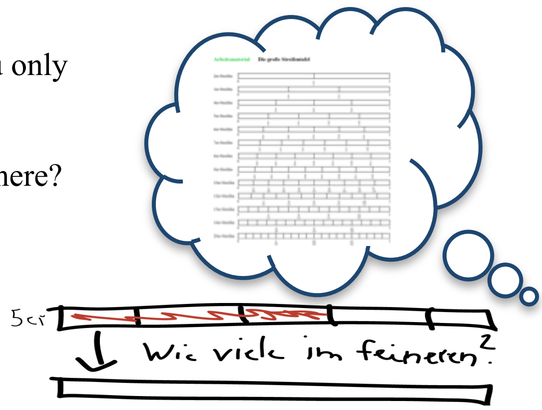


Figure 4. Mental practice for internalizing relational structures of equivalence

4. EVALUATION

Whereas laboratory design experiments and in-depth analyses offer good opportunities to understand the *situational effects* of the design (see preceding section), a quantitative evaluation of learning effects can better contribute to evaluate *the long-term efficacy* of the curriculum. That is why we conducted a first, rough assessment of effects in long-term classroom design experiments (2008-2010) with a standardized fraction test (see Table 1 for some items). We compared the performances of students in five classes (n=108) that have worked with our fraction curriculum in grades 5 to 7, with those of five neighbour classes (n=104) from the same schools that have used the usual textbook curriculum. This *pragmatic quasi-experimental sampling by neighbour classes* suggests the approximate comparability of treatment and control group with respect to general performances and socio-economic background (as these criteria were applied for composing classes in grade 5).

For comparing the long-term efficacy of two different fraction curricula, we measured students' fraction performance ten months after finalizing the curriculum with a *standardized fraction test* being adapted from Bruin-Muurling (2010) (more details in Prediger & Wessel, 2013). The test included 41 items which tried to limit specific training effects for the intervention group by a wide coverage of different contents: e.g., identify and draw fractions in part-whole and part-group models and on the number-line, order fractions and explain order in contextual or graphical representations, find equivalent fractions and explain, part of part-tasks, subtractions, operators.

Table 1 shows the *overall results* and those items on order and equivalence with most significant intergroup differences. The treatment group was significantly better (m = 23.49) than the control group (m = 19.52) in the whole fraction test, this difference increases enormously for the items on order and equivalence. Remarkable is also the difference in standard deviations, we interpret the higher homogeneity of the treatment group as a success in giving *also weaker students* an access to conceptual understanding.

Table 1. Comparing performances of treatment / control group for overall results and selected items

	Treatment Group	**Control Group**	**Significance**	
Sample size	n = 108	n = 104		
Mean of overall scores	m = 23.49	m = 19.52	T = 4.580	α < 0,001***
Standard deviation	SD = 5.389	SD = 7.146	F = 4.362	α < .05 **
Items with significant differences	Frequency of complete solutions			
Item 1a. Find fraction 3/4 from picture	99.1 %	86.5 %		
Item 5b. Compare 2/10 and 4/6	86.1 %	72.1 %		
Item 5d. Explain why 2/3 < 3/4	45.4 %	28.8 %		
Item 5f. Explain why 3/9 = 5/15	25.0 %	3.8 %		
Item 6a. Read on the number line 2/8	39.8 %	20.2 %		

5. CONCLUSION AND OUTLOOK

A design research project always aims at research results and design results: The central finding of the empirical *research* on students learning processes is that focussing on structural relations is an important condition for developing conceptual understanding of order and equivalence of fractions while relating different representations. Especially weaker students do not develop this focus automatically, but if they are fostered by suitable activities, they are able to.

Central results of the iterated *design* have been presented in three design strategies for implementing the design principle ‘focussing structural relations’, namely

1. constructing relevant structural relations by contexts and systematic variation,
2. embedding structural relations in a comprehensive visual model,
3. internalizing structural relations by mental practice.

The empirical snapshots (that could of course only show a minimal part of the large data set) illustrated how these design strategies can initiate learning processes. The qualitative insights into *situational effects* of the design were triangulated by a first rough evaluation of *efficacy* in a posttest-control-group design. The results show that the students who learned with our curriculum learned significantly more than the control group classes. However, this first rough evaluation has two important *methodological limits:* (1) the lack of fraction performance control in a pre-test in 2008 and (2) only partial control on the applied pedagogy in the long-term intervention in regular classrooms. Because of these methodological limits, we have started another evaluation study from 2012-2014. As the curriculum has further been developed, we hope to receive more robust results and perhaps even higher learning effects.

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