STUDENTS' MENTAL COMPUTATION STRATEGIES WITH RATIONAL NUMBERS REPRESENTED AS FRACTIONS

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The development of mental computation with rational numbers in fraction representation may help students to make sense of fractions and their operations, contributing to the development of rational number sense. This paper aims to analyze the development of students' mental computation strategies in fraction representation in grade 6, based on a teaching experiment focused on mental computation tasks with rational numbers involving the four operations and the discussion of students' strategies. The results show that, in the beginning of the study, the students use mainly instrumental strategies and, along the teaching experiment, they use more and more conceptual strategies.

Keywords: Mental computation, fractions, students' strategies, relational thinking, tasks.

INTRODUCTION

Mental computation with rational numbers tends to assume a marginal role in school curriculum and in teaching practice. Given children's difficulties in computing mentally and in learning rational numbers, it seems worthwhile to combine these two aspects. Systematic work with mental computation may help to develop computation skills (Heirdsfield, 2011). This may contribute to the development of students' strategies, reasoning, critical skills and number and operations sense. Assuming this perspective, the Portuguese mathematics curriculum for basic education (grades 1-9) (ME, 2007) suggests that mental computation with natural numbers must be developed from grade 1 on and later be extended to other numerical sets. However, despite such recommendation, most students have very poor ability in mental computation with rational numbers.

The aim of this paper is to analyze the development of students' mental computation strategies with rational numbers (in fraction representation) in grade 6, based on a teaching experiment focused on mental computation tasks with rational numbers involving the four operations and the collective discussion of students' strategies. More specifically, we want to know: (i) what strategies students use in mental computation tasks with rational numbers represented as fractions? (ii) what is the nature of these strategies (instrumental or conceptual)? and (iii) how these strategies evolve during the teaching experiment?

MENTAL COMPUTATION AND OPERATIONS WITH FRACTIONS

For many reasons, fractions are difficult to understand. A fraction is written using two numerals but represents only one number; although 2<4, we have 1/2>1/4; the same quantity may be represented by several fractions; the rules for multiplication of fractions are easy to accept (e.g., 1/2x2/5=2/10) but the rules for adding and subtracting fractions are not the most intuitive ones (1/2+1/4 is not 2/6) (Lamon, 2006) and these similarities and differences may induce students in misunderstandings about rational numbers and fractions. For Galen, Feijs, Figueiredo, Gravemeijer, Herpen, and Keijzer (2008), fractions must receive a special attention in elementary school, not only because knowledge about fractions is a starting point for understanding decimals and percent, giving meaning to them, but also because fractions play an important role in mental computation. Another aspect to underline is that we often think using fractions, even when they are not explicitly involved. For example, to estimate 72% of 600, we may associate 72% to 3/4 and consider half of 600 (i.e., 300) and a fourth of 600 (i.e. 150) obtaining 450, a close estimate. These authors call "network of relationships" the knowledge that students develop about different types of fractions and refer that this network does not develop just by practicing. They also consider that students' knowledge about fractions may be increasingly vast and formal, however this knowledge is not specifically associated to a context but to a given fraction. In their view, a student who understands that 3/4 is smaller than 4/5, is not necessarily ready to understand that 14/15 is less than 15/16.

To work with fractions in a significant way, students need to develop a sense of fraction. This requires to understand the relationship between numerator and denominator, and to realize that this relationship is crucial to determine the size and the value of a rational number. For Cramer, Wybeg and Leavitt (2009), a student who has number sense is reflective about numbers, operations and results and has flexibility in using comparison strategies and operations with numbers. As Cruz and Spinillo (2004) indicate, the use of benchmarks is particularly important to compare and operate with fractions. In their view, the benchmark half plays an important role in students' initial understanding of complex logical-mathematical concepts associated with rational numbers, and the use of this benchmark may facilitate the addition of fractions.

Empson, Levi, and Carpenter, (2010), consider that learning fractions is strongly supported if children develop relational thinking. The focus on relational thinking may help children to reason about the quantities involved. They consider that a child begins to think in a relational way about the quantities involved in a fraction, when he/she manages to relate the number of equal parts that he/she must share with the whole number of persons for which he/she must also distribute these parts. In their view, relational thinking allows to make sense of operations with fractions. When children understand fractions as a set of relationships, they are able to compose and decompose quantities for transforming and simplifying expressions. A key point in developing a child's understanding is reached when he/she start using relational thinking in his/her strategies to make repeated additions or subtractions of fractions more efficiently by applying fundamental properties of operations and equalities to combine quantities. A child who, to compute 8 groups of 3/8, thinks that if 8 groups of 1/8 is equal to unity, then 3/8 will be three units, is making a reasoning based on the commutative and associative properties of multiplication. Thus, the development of basic knowledge to think about fractions in an efficient way integrates knowledge of properties of natural numbers, their operations and their relationships and anticipates the generalization of algebraic quantities. The authors state that the use of algorithms to develop some fluency with operations with numbers is useful, but argue that if the development of students' relational thinking is supported, the knowledge about the generalization of the properties of numbers and operations becomes more explicit and may be the basis for the learning of algebra in the subsequent school levels, reducing students' errors and misconceptions. The students who, in their mental computation strategies use only known facts and memorized rules, i.e., instrumental strategies (Caney & Watson, 2003) did not develop relational thinking, in contrast with students who use their knowledge about numbers, their relationships and operations, who provide evidence of conceptual strategies. These students build an important conceptual foundation for learning algebra (Empson et al., 2010).

The use of relational thinking is implicit in the framework of Heirdsfield (2011), who stresses that students, to compute mentally, need to (i) understand numeration, i.e., the size and the value of numbers, (ii) recognize the effect of an operation on a number, (iii) know number facts, and (iv) make estimates to check the reasonableness of a solution. These concepts and understandings are based on notions of number sense (McIntosh, Reys & Reys, 1992), operation sense (Slavit, 1999), and on a set of numerical facts that students learn in school and use to create their personal strategies' based on relational thinking.

The collective discussions have an important role in supporting students in sharing and building a repertoire of mental computation strategies. Thompson (2009) suggests that, to develop students' mental computation, teachers must: (i) create a classroom environment where students feel comfortable talking about their strategies; (ii) listen attentively to students' explanations about their computation methods; (iii) reinforce students' positively as they use specific strategies; (iv) enhance students' knowledge about numbers and capacity to implement effective strategies; and (v) ensure that students go through different experiences to gradually develop increasingly sophisticated strategies. In summary, the focus on the development of students' relational thinking, through the use of different representations of rational numbers, benchmarks and shared strategies in collective discussions may be an asset to the learning of rational numbers.

RESEARCH METHODOLOGY

This study is qualitative (Bogdan & Biklen, 1982) with a teaching experiment design (Cobb, Confrey, diSessa, Lehere, & Schauble, 2003). The participants are a mathematics teacher and a grade 6 class with 20 children who previously worked with rational numbers in different representations (decimals, fractions, and percent) with the four operations. The teacher invited to participate in the study has a long experience in teaching mathematics. The first author was a participant observer in the classroom. Data collection took place through video and audio recording of classroom moments with mental computation tasks (February to May 2012). We give special attention to collective discussions where students share how they think to compute mentally. Audio recordings were made of the preparation and reflection meetings with the teacher as well as researcher's notes. The dialogues (audio and video recorded) that show students' mental computation strategies in collective discussions were transcribed to identify how these strategies evolved during the teaching experiment. We present examples of student's strategies in different task to illustrate the importance of the task design principles and the evolution of strategies in collective discussions. Students' mental computation strategies observed in data were categorized as instrumental or conceptual (Caney & Watson, 2003), and students' use of relational thinking (Empson et al., 2010) was analyzed.

THE TEACHING EXPERIMENT

This was the first time that these students were asked to compute mentally with rational numbers. In elementary school (grades 1-4) they did not develop mental computations with natural numbers. Before the teaching experiment, the students worked with rational numbers in different representations and operations with emphasis on algorithms. The teaching experiment tasks were discussed with the teacher. The anticipation of students' strategies supported the choice of tasks as well as the preparation of collective discussions. Class management, including collective discussions, were led by the teacher, with the first author making occasional interventions to clarify aspects related to students' presentation of strategies.

To design the tasks for the teaching experiment, we assume several principles related to (i) context; (ii) representations of rational numbers and use of benchmarks, (iii) students' strategies; (iv) cognitive demand (Henningsen & Stein, 1997); and (v) collective discussions. Concerning the context, we use tasks framed in mathematical terms and word problems because we assume that diversity is important to develop mental computations skills. We use three representations of rational numbers (decimals, fractions and percent), in a progressively integrated way and taking advantage of benchmarks. We favor tasks that may promote the development of mental computation strategies (based on Caney & Watson, 2003) and tasks with different levels of cognitive demand to engage students in different kinds of reasoning (Henningsen & Stein, 1997) (one involving numbers with which the students may operate easily, and another where they have to use numerical relationships). And, finally, we regard collective discussions as a fundamental aspect in the development of mental computation strategies because they provide students with the opportunity to share ideas, reasoning and to improve their language by communicating mathematically, as they try to explain their reasoning.

The teaching experiment includes ten mental computation sets of tasks with rational numbers (with the four operations) to carry out each week for about 15-20 minutes at the beginning of a mathematics class. Of these ten tasks, seven are mathematical exercises that require students to compute the result of an operation or the value that makes a given equality true; two sets of tasks have four word problems each; and a set of tasks has a mix of exercises and problems. The rational number representation and the problems were related to the topics that students were working in class. When they were working with algebra, they computed with fractions, when working with volumes, they computed with decimals, and when working with statistics they computed with percent, fractions, and decimals. This is an aspect that we consider new since it allow mental computation to be seen in an integrated way and rational numbers learning is extended in time and thus allow connections with other mathematical topics intentionally.The questions are presented using a timed PowerPoint, allowing 15 seconds for each exercise and 20 seconds for each problem. The students recorded their results on paper. After finishing the first five exercises, there was a collective discussion of students' strategies. Then, the students solved the second part of the task and that was followed by another collective discussion. These discussion moments allowed the students to reflect on how they think, what strategies they use, and what errors they make.

In the teaching experiment, the students solved two mental computation tasks in mathematical terms using only fractions (addition/subtraction and multiplication/ division), three tasks in mathematical terms with fractions, decimals and percent (with the four basics operations) and word problems where fractions appeared in combination with the other two representations. However, several times the students' used fractions in tasks that present only operations with decimals or percent, especially involving benchmarks as 0.25 or 75%. In table 1 we present the tasks (exercises and problems) where we used fractions that students solved mentally throughout the teaching experiment and that we analyze in this paper.

<table>
<tr><td>Task 1
3/4-1/2
3/6 + ?=1</td><td>Task 2
4/6:2/6
?x5/6=1/6</td><td>Task 3
0.75:1/4
8/10-0.2</td></tr>
<tr><td>Task 6
A tank has a capacity of 22.5l of water. How many buckets of 1/2l are needed to fill the tank completely?</td><td>Task 8
1/3 of 1/3
1/5 of ?=8</td><td>Task 9
4/6 x 6/7
2/3 x ? = 1</td></tr>
<tr><td colspan="3">Task 10
Every day, 400 students eat lunch at Johns' school.
Of these students, 3/4 always eat soup. How many students eat soup?</td></tr>
</table>

Table 1. Example of mental computation tasks using the fraction representation.

The tasks take into account several aspects important in mental computation. They invite the use of benchmarks such as 1/2, 1/4 and 3/4, as well as the use of halves. We also used less common fractions such as 1/3 or 4/6 expecting that the students would apply the knowledge developed with benchmarks in dealing with them. We start with addition and subtraction because these operations are the first that students study in grade 5, and then we use multiplication and division that students learn in grade 6. The operations involve fractions with the same denominator or denominators involving multiples so that students may use equivalence. In the case of word problems, the context is related to the mathematical topics that the teacher was working in the classroom (e.g., statistics, measurement, comparison of rational numbers and percent). We created problems that could lead to expressions similar to those discussed during the teaching experiment.

STUDENTS' MENTAL COMPUTATION STRATEGIES WITH FRACTIONS

In mental computation with rational numbers in fraction representation, students use mostly instrumental strategies by applying known facts and memorized rules (Caney & Watson, 2003). However, it is possible to identify the development of conceptual strategies along the teaching experiment. In these conceptual strategies, students' use equivalence, numerical relationships and properties of operations or mixed strategies combining instrumental and conceptual features. In tasks 1 and 2, involving first addition and subtraction, and later multiplication and division, students' strategies were clearly instrumental. To compute 1/2+1/2, Elsa explains how she applied the rule of addition of fractions: "As the denominators are the same, I put the same denominator and I added the numerator. (...) It is equal to 1. [It is] 2/2 that is equal to 1". Marta uses a known numerical fact to compute the result of the same expression. She quickly made "half plus a half" knowing that this gives 1. She shows that she memorized some numerical facts (as two halves forming a unit), and used it to do the computation. To compute 3/4x2/3, Bruno explains how he used a memorized rule: "I applied a "cutting law" ["if the numerator of a fraction is equal to the denominator of another fraction, I may cut both of them"], and it gives me 2/4. To compute 4/8:1/2 Rita used the invert-and-multiply algorithm in the division of fractions, as she explains: "I inverted the 2 with the 1 and did 4x2 that gives 8. So this yields a unit [8/8] and I wrote 1". The numerical expressions presented above have a low cognitive demand and these where the first tasks from the teaching experiment. These two aspects may influence the nature of the strategies used by students.

In task 3, some students continue to use instrumental strategies. However, conceptual strategies begin to emerge. That may have been influenced by the collective discussions of the previous tasks and also because fractions appear in combination with decimal representations. For example, João uses a conceptual strategy to compute 3/4+0.5. He explains that: "I took 1from 3/4 [took 1/4] and, as 2/4 add with 1/2 give 1, in the end I add what I had taken, and give 5/4". This student decomposed 3/4 in 2/4+1/4, and added two halves to obtain a unit (2/4+1/2=4/4) without calculating the same denominator. Finally, he added the amount taken from 3/4 and obtained 5/4. The use of two representations (fraction and decimal) and the benchmark made João realize that it was useful to change one representation.

A discussion moment regarding 8/10-0.2 shows how the interaction between teacher and students and between students is important to share strategies, emphasize equivalence, lead students to find new strategies and increase the repertory of strategies. The discussion starts with Rita:

Rita: It gave me exactly six tenths.

Teacher: How did you think?

Rita: When I saw the... That there was two tenths, I immediately thought that it was a fraction equivalent to 1/5. And then I multiplied the numerator and the denominator and it gave me two tenths [she added numerators and maintained the denominators].

Teacher: I do not want to comment. Elsa.

Elsa: I did... It gave me the same result but I did it in a different way from Rita. As there was two tenths I put two tenths in fractions and as the denominators were the same, I left the denominators and subtracted 8-2. That gave me 6 and then I put the same denominator. It gave me 6/10.

Teacher: So, Rita do you think that we have here another step? It is similar to your strategy but it is easier, no?

Rita: Yes, maybe.

Teacher: Maybe? You had to turn in to a fifth [0.2 in to 1/5] and you had to turn 1/5 again, she saw that it is two tenths and maintained the same denominators. Who made differently. Ana?

Ana: When I saw there the eight tenths I put it in a irreducible fraction and it gave me 4/5. Subtracting 1/5, it gave me 3/5.

Teacher: 3/5 is the same that…?

Ana: Six tenths.

Teacher: Six tenths, Funny that. Someone did it different? I did. Someone? Nobody…

João: As the 8/10 as a denominator 10 I just put 0.8 or eight tenths.

Teacher: You did not, you did it now!

João: Yes I did.

All the students used conceptual strategies changing representations and using equivalent fractions. Then, they use instrumental strategies applying the addition of fraction algorithm. After the teacher request, João indicates a new strategy.

This episode shows the importance of collective discussions to promote sharing strategies. Discussions lead students to make conjectures, to reflect on why they made some errors and to validate strategies and errors. We consider that this is useful in developing students' use of conceptual strategies. In task 6, to solve the problem presented in table 1, Eva refers that: "It gave me 45 buckets. I multiplied 22.5 by 2 (…) Because [we have] 1/2, to get the unit we have to add 0.5 twice, so, I multiply by 2". This student thinks in a half using fraction and decimal representations, showing no confusion, and establishes a relationship between the capacity of the bucket and the capability of the tank ("with a bucket of 1l I can take 22.5 buckets. With a bucket with half of capacity, I have to take the double number of buckets"). Eva does not use the invert-and-multiply algorithm. She uses a conceptual strategy based on numerical relationships that gives meaning to this rule (Newton & Sands, 2012).This word problem promoted the emergence of relational thinking in Eva's strategy.

In task 8, some students continue to use instrumental strategies (such as Duarte), but also conceptual strategies (such as Maria). To compute the same operation, 1/3x1/3, Duarte explains:"It gives me 1/9. I multiplied denominators and numerators", using the algorithm of multiplication of fractions. Maria shows to have number sense by explaining that 1/3 of 1/3 it is 0.111… This student describes her reasoning: "I know that 1/3 it is 0.333… dividing by 3… is like a kind of the multiplication table by 11 (…) 11x1=11, 11x2=22… here it is 33:3 which gives 11". She operates based on the knowledge of the 11 multiplication table. Maria used a strategy based on numerical relationships.

In task 9, to compute 2.2-?=1/5, José used a conceptual strategy and to compute 2/3x?=1, Pedro used an instrumental strategy. José changed the representation: "We transform 1/5… it is 0.2" and used a property of subtraction "So, 2.2-0.2 gives 2" – to get the subtractive, he subtracted the remainder from the additive. Pedro used a known numerical fact and says that: "It is 3/2. They are inverse fractions". This student memorized that the multiplication of a fraction by its inverse gives the unit.

In task 10, to solve the problem that we presented in table 1, Ana used an instrumental strategy applying the rule of multiplication of fractions by a natural number: "400x3 gives 1200 [and] dividing by 4 it is 300". João uses equivalence showing a conceptual strategy: "300/400 is equivalent to 3/4. As 400 was the unit, it had to be 300".

CONCLUSIONS

The analysis of the seven mental computation tasks carried out with rational numbers represented by fractions shows that in the first two tasks students' strategies were mostly instrumental (Caney & Watson, 2003). They used known numerical facts or memorized rules to operate with fractions. In addition and subtraction, they computed the same denominator and added the numerators, in multiplication, they multiplied numerators and denominators and, in division, they used the invert-and-multiply algorithm (as Elsa, Marta, Bruno and Rita did). From task 3 on, conceptual strategies begin to emerge (Caney & Watson, 2003) based on numerical relationships, equivalence and properties of operations, indicating an increasing use of relational thinking (Empson et al., 2010) (as João, Eva, Maria and Duarte did). Eva's strategy is particularly useful for making sense of the "invert-and-multiply" rule.

The choice of tasks and the classroom collective discussions contributed to this evolution. Using tasks in two contexts allowed students to diversify their strategies. In tasks in mathematical terms (such as tasks 1 and 2), the students tend to use instrumental strategies and in problems (such as tasks 6 and 10) they tend to use complex strategies based on relational thinking. From task 3 on, students have to compute with fractions together with decimals which suggests the change of representation and the use of equivalence, an important strategy (Caney & Watson, 2003). Also, the use of benchmarks during the teaching experiment helped students to make sense of rational numbers (Galen et al., 2008) since they used frequently the change of representation and equivalence. The variation of cognitive demand of tasks, in different contexts, provided students with opportunities to develop increasingly complex personal strategies (as Eva did). Furthermore, collective discussions promoted sharing of strategies, contributing to increasing the students' repertoire of strategies and validation processes, improving their capacity to conjecture and to justify how their mental computation processes as we illustrated in the discussion that emerged from task 3. These interactions where important to help students to construct conceptual understanding (e.g., by discussing equivalences) about rational numbers and this is reflected in the use of more conceptual strategies along the teaching experiment.

This study shows that it is possible to develop students' mental computation processes by carefully designed teaching. In this teaching experiment, tasks were constructed considering principles that proved to be fruitful and involved the three representations of rational numbers simultaneously. The teaching experiment was extended in time, allowing the work in mental computation with rational numbers to establish connections with work on algebra, geometry and statistics. In addition, this study emphasizes the importance of the collective discussions as a way to develop student's strategies. Therefore, we show that systematic work in mental computation may promote the development of personal strategies increasingly based on numerical relationships and properties of operations, enabling students to develop relational thinking that will be useful in learning algebra.

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