DETERMINING SPECIALISED KNOWLEDGE FOR MATHEMATICS TEACHING

J. Carrillo, N. Climent, L.C. Contreras & M.C. Muñoz-Catalán[1]

University of Huelva, Spain

ABSTRACT

Drawing on the work of Deborah Ball and collaborators in the field of Mathematical Knowledge for Teaching (MKT), we draw attention to several areas of difficulty in applying this framework to actual samples of mathematics lessons, due to a tendency for the subdomains that make up the model to overlap. Tackling these shortcomings by viewing all mathematics teachers' knowledge as specialized has led us to reinterpret and rename these subdomains in what can be considered a reformulation of MKT.

***Keywords**: mathematical knowledge for teaching, teachers' specialized knowledge, mathematics teachers' knowledge.*

INTRODUCTION

One of the benefits of research into teachers' knowledge – in our case relating to mathematics teaching – is ascertaining desirable elements and characteristics that can be taken as reference points when working with teachers. From the many such characterisations of knowledge developed in the last two decades (Bromme, 1994; Rowland, Turner, Thwaites, & Huckstep, 2009), the theory of *Mathematical Knowledge for Teaching* (Ball, Thames, & Phelps, 2008) has proved to be especially powerful in describing the knowledge required by teachers in their practice, underlining its ties with mathematics while at the same time considering other elements involved in the teaching process (e.g. the pupils and their learning, and the curriculum) and the connections between them. What is more, *Mathematical Knowledge for Teaching* (MKT) has pioneered consideration of mathematical knowledge from the point of view of teaching, including knowledge of the structure of the subject, the rules governing how it works, and careful thought about the contents and their relations. In this respect, it seems to us that the purpose of MKT is that it should be an analytical tool for studying teachers' knowledge, as opposed to a model of such knowledge itself. On the other hand, we are aware that the authors'

---

[1] This article represents part of the studies into teachers' knowledge by the SIDM group (from the Spanish 'Research Seminar into Mathematics Education') based at the University of Huelva, Spain. It comprises the following researchers: José Carrillo (coordinator), Nuria Climent, M. Cinta Muñoz-Catalán, Luis C. Contreras, Miguel A. Montes, Álvaro Aguilar, Dinazar I. Escudero, Eric Flores and Enrique Carmona (University of Huelva), Pablo Flores, Nielka Rojas and Elisabeth Ramos (University of Granada, Spain), C. Miguel Ribeiro, Rute Monteiro and C. Susana dos Santos (University of the Algarve, Portugal), Leticia Sosa and José L. Huitrado (University of Zacatecas, Mexico), and Emma Carreño (University of Piura, Peru).

description of teachers' knowledge is partial, omitting other equally important dimensions, such as teachers' beliefs and knowledge not specifically related to mathematical issues such as class management.

MKT has meant a significant advance in attempts to characterise mathematics teachers' knowledge domains. The most significant contributions are probably the differentiation of the subdomains *specialized content knowledge* (SCK), *common content knowledge* (CCK), and *horizon content knowledge* (HCK), within Shulman's (1986, 1987) classic *content knowledge*, in addition to the subdomains *knowledge of content and students* (KCS) and *knowledge of content and teaching* (KCT), within *pedagogical content knowledge*. However, as the authors themselves recognise, the new subdomains do not in practice mean an exhaustive classification of a teacher's knowledge, and often it is difficult to know whether an excerpt of classroom practice is unambiguously illustrative of one of these subdomains or rather the intersection of two or more of them. What is more, the very differentiation between SCK and CCK leaves the system open to the possibility that all teachers' knowledge is to some extent specialised.

It is precisely such dificulties in applying MKT to our studies that we wish to highlight in this paper, at the same time suggesting a reformulation of the framework from a perspective that does indeed regard all teachers' knowledge as specialised.

OUR EXPERIENCE WITH MKT

In our work on mathematics teachers' knowledge using the MKT framework (Sosa, & Carrillo, 2010; Figueiras, Ribeiro, Carrillo, Fernández, & Deulofeu, 2011; Ribeiro, & Carrillo, 2011a, b; Climent, Romero, Carrillo, Muñoz-Catalán, & Contreras, in press), we have identified various difficulties which have led us to raise questions about the model. The chief shortcomings were recognised by Ball and associates in their 2008 work, and concern *specialized content knowledge* (SCK) and *common content knowledge* (CCK [1]). Essentially, there are two related problems [2]:

1) The difficulty in deciding where CCK ends and SCK begins, as a result of the very definition of CCK. In brief, CCK is defined as that knowledge held by anybody educated to the corresponding level under analysis (Ball *et al*., 2008). In this way, although the model of MKT is based on observation, in order to decide whether the knowledge underpinning a teacher's action during a teaching episode corresponds to CCK or not, we need to compare it with the hypothetical knowledge of someone at a hypothetical level of education, without knowing anything about the educated person's practice or their typical knowledge, but instead, a compendium of desirable knowledge drawn from various curricula. Hence, for example, it is not clear whether CCK or SCK is invoked when explaining why the same denominator is needed for adding or subtracting fractions (but not in the case of multiplication or finding the quotient, although this might lead us to an alternative algorithm). Deciding

whether such knowledge is typical of a well-educated individual involves a large degree of speculation. It therefore occurs to us more reasonable to define CCK intrinsically, that is, referring exclusively to mathematical knowledge itself, without reference to other professions or qualifications.

2) The difficulty in demarking SCK from HCK, and SCK from KCS, again as a result of the definition of SCK. SCK is understood as a way of thinking about mathematics which occurs only when considered as something to be taught. However, it is sometimes difficult to determine whether this reflection refers to the relations between the item to be taught and others (HCK) or to the learning of the item (KCS). In this case, we can consider the example of the commutative property in relation to different objects. First, we will consider this property in relation to the addition and multiplication of natural numbers. Although both operations fulfill this property for this particular numerical set, from the point of view of their meaning, we can say that addition is semantically commutative, but not multiplication in general (adding or uniting 2 elements and then 3 is the same whichever the order; considering 3 groups of 2 elements is not the same, however, as considering 2 groups of 3 elements). This subtle difference affects how each case is perceived, and relates to how each is learnt. Now we will consider the property in relation to multiplying matrices. In this case, commutability does not generally occur, except in the case of square matrices in which the operation can be done either way (although these matrices do not fulfill the commutative property either). This fact differentiates the multiplication of matrices from that of numbers, and knowledge of this difference implies associating both contexts, which we would argue forms part of HCK. Additionally, it provides a mathematical explanation for a common student error in multiplying matrices, which associates it with KCS.

We have found, then, problems in the demarcation of the subdomains, in which respect we concur with the impression of other authors (Silverman, & Thompson, 2008). There is a need, we feel, to define the subdomains in a slightly different way, more appropriate, we would say, to teachers' knowledge regarding teaching mathematics. At the same time, we have tried to see to what point SCK permeates, or is included within, other subdomains, thus emphasising the valuable contribution it has made to the MKT model.

It is important at this point to give due recognition to the development of the MKT model by Ball and her collaborators, although at the same time the difficulties noted above lead us to think that it would be more appropriate to alter the focus of teachers' knowledge so that, on the one hand, it can be better understood, and on the other, its contents can be better discerned.

The difficulties in demarcation suggest the need to look more closely at MK (ie, 'mathematical knowledge' [3], that is the left hand side of the MKT model) and to progress towards defining and delimiting CCK, SCK and HCK. This is the work which we have undertaken and which we would like to present in this paper, along with the reformulation of the subdomains pertaining to *pedagogical content knowledge* (PCK, the right hand side of the MKT model). Throughout, we have been guided by two premises. First, we have not limited ourselves to merely observing episodes of classroom practice, but have proposed a sound theoretical model, which can be subsequently tested in practice (observations), especially longitudinal studies combining classroom observation and shared reflection. Secondly, we have remained open to the possible restructuring of the MK domain, and the potential for new or different subdomains, and even the possibility of the subdomains of PCK being affected. Moreover, this model is designed to reflect teachers' beliefs about mathematics and its teaching and learning. Although the role of these beliefs in the model is not the focus of this paper, the fact of its inclusion marks a divergence from the MKT model.

SPECIALISATION AS A GENERAL FEATURE OF MATHEMATICS TEACHERS' KNOWLEDGE: MATHEMATICS TEACHERS' SPECIALIZED KNOWLEDGE (MTSK)

What interests us as researchers and trainers within the area of Mathematics Education is the extension of teachers' professional knowledge linked to mathematics as the focus of the teaching-learning process, the recognition of which was one of the chief contributions of the work of Shulman (1987). For their part, the research team headed by Ball (Ball *et al*, 2008; Ball, & Bass, 2009) outline the mathematical knowledge within the specialised area, and it is precisely this mathematical character which causes problems when it is applied to *pedagogical content knowledge*.

We attempt to focus the specialisation of mathematics teachers' knowledge from another perspective. Instead of talking about 'specialised content knowledge' (as a part of teachers' knowledge), we talk about 'mathematics teachers' specialised knowledge' (*MTSK*). We try to distance ourselves from the idea of mathematical knowledge for teaching and think of mathematics teachers' knowledge that makes sense only to them (in which, therefore, the specialised nature defines all knowledge under consideration).

The specialisation of *MTSK* should allow it to be differentiated from general pedagogical knowledge (knowledge of pedagogy and general psychology, which also forms part of mathematics teachers' professional knowledge), from the specialised knowledge of teachers of other disciplines, and the specialised knowledge of other mathematics professionals. In other words, it is specialised in respect of mathematics teaching.

We have reconsidered the content of this knowledge from this perspective, basing ourselves on the domains of MKT and on our beliefs concerning what we consider desirable as the content of mathematics teachers' specialised knowledge in each subdomain. The outcome is that we propose to eliminate the reference to 'common content knowledge' from the domain of mathematical knowledge (given that our interest lies only with knowledge in relation to mathematics teachers; for example, we believe that teachers should possess not only the knowledge of *how* but the knowledge of *why*, and the students too, see Flores, Escudero, & Carrillo, 2012). As a result, 'specialised content knowledge' ceases to be necessary and 'horizon content knowledge' broadens its scope (resulting in two related subdomains). With regard to 'pedagogical content knowledge', we have renamed and reinterpreted KCS, KCT y KCC, recallibrating them to what we believe is their content.

We present this new proposal in more detail below and in a visual display (Fig. 1).

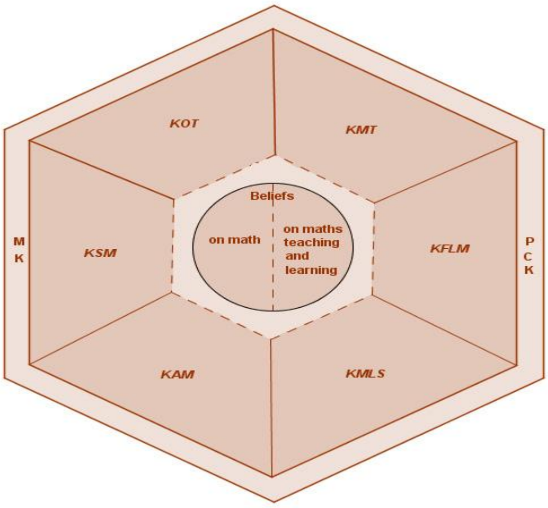

Fig. 1: Chart of *MTSK*

Elements of *MTSK* refering to Mathematical Knowledge (MK)

a) Knowledge of topics (*KOT*)

This includes the knowledge of mathematical concepts and procedures along with the corresponding theoretical foundations. We can say that all knowledge considered desirable for a pupil to be in possession of at any particular level [4] would form part of the teacher's CCK at this level, including a certain degree of formalisation or vision of the content from a somewhat higher viewpoint (for example, knowing that the property of commutability represents a more technical explanation of the fact that the order of addends in an addition sum does not affect the result).

b) Knowledge of the structure of mathematics (*KSM*)
Building on Ball, & Bass' (2009) description of horizon knowledge, we consider two elements of MK relating to the structure of the discipline (this subdomain) and the ways of proceeding in mathematics [5] (the next subdomain).
The first of these elements, knowledge of the structure of the discipline, includes knowledge of the main ideas and structures, such as knowledge of properties and notions relating to specific items being tackled at any moment, or the knowledge of

connections between current topics and previous and forthcoming items. It implies seeing the content in perspective, basic mathematics from an advanced point of view, and advanced mathematics from a basic point of view. Also included is the idea of increasing complexity, as explained in Montes, Aguilar, Carrillo, & Muñoz-Catalán (2012, in this volume).

c) Knowledge about mathematics (*KAM*)

The second of these elements refers to ways of proceeding in mathematics. It includes knowledge of ways of knowing and creating or producing in Mathematics (syntactic knowledge), aspects of mathematical communication, reasoning and testing, knowing how to define and use definitions, establishing relations (between concepts, properties etc), correspondences and equivalences, selecting representations, arguing, generalising and exploring. Knowledge about relations or connections between concepts, pertaining to *knowledge of the structure of mathematics*, should be distinguished here from knowledge about how such relations are established.

Defined in this way, MK extends over the full range of mathematical knowledge, covering the whole universe of mathematics, comprising concepts and procedures, structuring ideas, connections between concepts, the reason for, or origin of, procedures, means of testing and any form of proceeding in mathematics, along with mathematical language and its precision. The denomination *KOT* emphasises that the subdomain is defined in purely mathematical terms, and this we think makes it clearer that *knowledge of topics* and *knowledge of the structure of mathematics* form a complex system. At the same time, this way of defining *KOT* avoids the somewhat mechanical slant which the definition of CCK was prone to, as sometimes was the knowledge *of* mathematics (in the sense of knowledge *of* and *about* mathematics, Ball, 1990).

Returning briefly to one of the examples above, knowing that the product of matrices is not commutable pertains to *KOT*; knowing that in this sense it is different from the multiplying natural numbers would pertain to *knowledge of the structure* (as it means taking a basic viewpoint to the multiplication of matrices, like multiplying numbers) and knowing that the pupils believe that the product of matrices is commutative because they extrapolate this property from multiplying numbers (which they learn at school) would form a part of *pedagogical content knowledge* (as we will now explain). If we take the point of view of *specialized content knowledge* within the MKT model, reflection about specialisation in both contexts is a reflection about the specific content of the act of teaching, for which reason we consider it as SCK, which, as we have noted above, results in overlap with *horizon knowledge* and *knowledge of content and students*.

Elements of *MTSK* refering to Pedagocical Content Knowledge (PCK)

d) Knowledge of Features of Learning Mathematics (*KFLM*)

KFLM derives from the teacher's need to understand how pupils think when faced with mathematical activities and tasks, the same as KCS in Ball's model. It is important that the teacher is aware that the pupils may have problems with a particular topic. This awareness is fed by the teacher's general knowledge of the topic and by their familiarity with the pupils. This subdomain encompasses a range of knowledge, including (and not, we believe, explicitly included in KCS) theories or models of how students learn mathematics (for example, the process which takes pupils from action to schema according to the APOS perspective –Asiala, Brown, DeVries, Dubinsky, Mathews, & Thomas, 1996). It is not a question of knowing these theories or perspectives, but rather their significance, that is to say, what these theories contribute to describing the process of learning mathematics. *KFLM* is not mathematical knowledge, although the teacher needs to have a background in mathematics in order to understand it and put it to use. KCS refers to content and students, while *KFLM* is concerned with how mathematics is learned, that is, with identifying the features of mathematics learning.

e) Knowledge of Mathematics Teaching (*KMT*)

KMT is not mathematical knowledge either, though it does require it. It is the kind of knowledge which allows the teacher to choose a particular representation or certain material for learning a concept or mathematical procedure, and which allows them to select examples or choose a textbook, in much the same kind of way as Ball's KCT. We would underline here (encapsulated in the name of the subdomain) the integration of mathematics and teaching, in that it is not a question of mathematical knowledge on the one hand and teaching knowledge on the other; pedagogic knowledge is not included here in the context of mathematical activities, but rather only that in which the mathematical content constrains the teaching. In *KMT* we locate knowledge of resources from the point of view of their mathematical content or the knowledge of approaching a structured series of examples to help pupils understand the meaning of a mathematical item.

f) Knowledge of Mathematics Learning Standards (*KMLS*)

KMLS concerns knowledge of curricular specifications, the progression from one year to the next, conventionalised materials for support, minimum standards and forms of evaluation, in the same way as KCC does in Ball's model. However, *KMLS* seeks to extend knowledge of learning objectives and standards beyond those deriving from the institutional context of the teacher. We include objectives and measures of performance developed by external bodies such as examining boards, professional associations and researchers, thus adding an element of assessment and evaluation drawn from the appropriate educational agencies.

FINAL COMMENTS

With *MTSK* we have intended to focus solely on mathematics teachers' specific knowledge with respect to teaching the subject, eliminating any reference to a common core of knowledge shared with others who make use of mathematics. *Knowledge of topics* and *Knowledge about mathematics* are shared by all mathematicians, as is to a certain extent *Knowledge of the structure of mathematics*, though not to the degree of familiarity required by teachers. Conversely, *KMT*, *KFLM* and *KMLS* are exclusive to teachers.

With MKT the focus was on the class as a whole, including pedagogical concerns (KCT, KCS) to an extent that the framework might be applicable to other disciplines, but in so doing it shifted away from mathematics and its core essence. In contrast, rather than consider the class as a whole, we aim to consider mathematics as the hub of *MTSK*, around which this new framework offers different ways of viewing the mathematics which the teacher knows and uses. We refer not only to mathematics in itself, but to reflections about mathematics that a teacher establishes by interacting with it in their daily practice, out of which aspects of mathematics pedagogy inevitably arise (*KLSM*, *KMT*). MKT concerns the educational circumstances constraining the teacher: recognising the causes of error, using powerful examples, identifying incorrect definitions in textbooks, etc. In contrast, *MTSK*, by virtue of being designed to encapsulate teachers' specialised kowledge, focuses its attention on mathematical content and, with greater precision, on the different ways of fully engaging with mathematical content when teaching.

In this paper we have endeavoured to discuss the defining features of the subdomains comprising MKT and suggest an alternative model based around it. We propose that this model (and others) be conceived of, and brought into play, as a kind of researcher's kit which helps them to avoid a prescriptivism which might impede understanding the phenomenon under scrutiny. Studying *MTSK* using this kit will enable our knowledge of its categories and subdomains to be reinforced.

At the same time, it would be interesting to pursue another line of research attempting to situate the model within a theoretical framework, in which, amongst other things, we would have to explicitly present our grounded position regarding how we understand teaching and learning mathematics, teacher training (both initial and in-service), our mathematics beliefs, the utility of models and other analytical tools, the purpose of our research, and the role of the subjects being studied/participating in that research, especially in relation to the researchers and the carrying out of the research itself.

In our work group we are drawing up research projects into mathematics teachers' knowledge in terms of different topics and different kinds (some structuring, such as the notion of infinity, and others more local, such as the concept of a polygon), and focusing on different dimensions of *MTSK*. Some of these projects involve experienced teachers (occasionally in the context of professional development), others with novice teachers, and others with students' teachers. Different educational phases are also involved, and likewise different stages of the teaching-learning process (exemplification, the introduction of concepts or procedures, designing tasks, and making decisions in class). Our aim across the board is to explore the limits and potential shortcomings of our proposal for *MTSK*, and to refine it further.

It stripes us that a better specification of desirable professional knowledge for a mathematics teacher from research is expecially important in contexts of professional development, particularly in collaborative situations, where the group itself is at liberty to decide what to study and reflect on (in terms of professional practice, for example), *MKTS* being one such possibility. It is not a question of having available a model which can be gradually assimilated, so much as having this model available as a point of departure for shared reflections forming the platform on which to design the group's collaborative work (Carrillo, & Climent, 2011).

Acknowledgements

The authors are members of the research project "Mathematical knowledge for teaching in respect of problem solving and reasoning" (EDU2009-09789EDUC), funded by the Ministry of Science and Innovation in Spain.

NOTES

1. We use the acronymns coined by Ball and collaborators themselves; in addition to those above, KCT for Knowledge of Content and Teaching, KCS for Knowledge of Content and Students, and likewise HCK for Horizon Content Knowledge and KCC for Knowledge of Content and Curriculum.
2. For further explanation, as well as examples of the difficulty in applying MKT, see Flores, Escudero, & Carrillo (2012, this volume).
3. Mathematical knowledge is understood as Shulman's subject matter knowledge. We use MK instead of SMK to avoid confusion with SCK or SMK in reference to specialised content or mathematical knowledge, respectively.
4. From a conception of school mathematics in which the pupils also learn the 'whys' of procedures and the reasons for certain concepts (for example, why fractions with the same denominator are needed to add fractions, but not to multiply them).
5. Ball, & Bass (2009, p.6) mention four elements constituting HCK: "a sense of the mathematical environment surrounding the current "location" in instruction; major disciplinary ideas and structures; key mathematical practices; and core mathematical values and sensibilities". In our case, we have considered the first three elements, given that values and sensibilities means introducing a different kind of element from the rest of the components of the model.

REFERENCES

Asiala, M., Brown, A., DeVries, D. J., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A Framework for Research and Curriculum Development in Undergraduate Mathematics Education. In J. Kaput, A. Schoenfeld, & E. Dubinsky (Eds.), *Research in Collegiate Mahematics Education II, CBMS Issues in Mathematics Education* (Vol. 6, pp. 1-32). Washintong DC.: American Mathematical Society y Mathematical Association of America.

Ball, D.L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *Elementary School Journal*, 90, 449-466.

Ball, D.L., Thames, M.H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59 (5), 389-407.

Ball, D. L., & Bass, H. (2009). *With an eye on the mathematical horizon: Knowing mathematics for teaching to learners' mathematical futures*. Paper presented at The 2009 Curtis Center Mathematics and Teaching Conference.

Bromme, R. (1994). Beyond subject matter: A pshychological topology of teaqchers' professional knowledge. In R. Biehler, R. Sholz, R. Strââer, & B. Winkelmann (Eds.), *Didactics of Mathematics as a Scientific Discipline* (pp. 73-88). Dordrecht: Kluwer Academic Publisher.

Carrillo, J., & Climent, N. (2011). The development of teachers' expertise through their analysis of good practice in the mathematics classroom. *ZDM*, 43, 915-926.

Climent, N., Romero, J.M., Carrillo, J., Muñoz-Catalán, M.C., & Contreras, L.C. (in press). Qué conocimientos y concepciones movilizan futuros maestros analizando un vídeo de aula. *Relime*.

Figueiras, L., Ribeiro, C.M., Carrillo, J., Fernández, S., & Deulofeu, J. (2011). Teacher's advanced mathematical knowledge for solving mathematics teaching challenges: a response to Zazkis and Mamolo. *For the Learning of Mathematics*, 31(3), 26-28.

Flores, E., Escudero, D.I., & Carrillo, J. (2012). *A theoretical review of specialised content Knowledge*. Manuscript submitted for publication (CERME 8).

Montes, M., Aguilar, A., Carrillo, J., & Muñoz-Catalán, M.C. (2012). *MSTK: from Common and Horizon Knowledge to Knowledge of Topics and Structures*. Manuscript submitted for publication (CERME 8).

Ribeiro, C. M., & Carrillo, J. (2011a). Knowing mathematics as a teacher. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of Seventh Congress of European Society for Research in Mathematics Education, CERME 7* (pp. 2817-2826). Rzeszów, Polonia: ERME.

Ribeiro, C. M., & Carrillo (2011b). Discussing a teacher MKT and its role on teacher practice when exploring data analysis. In B. Ubuz (Ed.), *Developing Mathematical Thinking, Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 41-48). Ankara, Turkey: PME.

Rowland, T., Turner, F., Thwaites, A., & Huckstep, P. (2009). *Developing primary mathematics teaching*. London: SAGE.

Shulman, L.S. (1987). Knowledge and teaching: Foundations of a new reform. *Harward Educational Review*, 57(1), 1-22.

Silverman, J., & Thompson, P. W. (2008). Toward a framework for the develoment of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education*, 11, 499-511.

Sosa, L., & Carrillo, L. (2010). Caracterización del conocimiento matemático para la enseñanza (MKT) de matrices en bachillerato. In M.M. Moreno, A. Estrada, J. Carrillo, & T.A. Sierra (Eds.), *Investigación en Educación Matemática XIV* (pp. 569-580). Lleida, Spain: SEIEM.