MATHEMATICS IN PRE-SERVICE TEACHER EDUCATION AND THE QUALITY OF LEARNING: THE PROCEPTUAL DIVIDE

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This paper presents two episodes of an exploratory study on a prototype of a mathematics curriculum for pre-service teacher education. The focus here is on the way in which students may operationalize mathematical concepts in rational arithmetic thus improving the mathematical quality of their learning. It is claimed in the major research that the curriculum should offer a mathematical matrix strong and flexible enough to enable them to manipulate and create the conditions to teach mathematics with quality as a generalist teacher. The main claim in this paper is the difference between those who relate and compress arithmetic procedures and those who remain in rote learning step-by-step arithmetic procedures, that is called the proceptual divide as showed.

Keywords: *Arithmetic's, curriculum, pre-service teacher education, proceptual divide, SOLO levels.*

INTRODUCTION

This paper presents an exploratory study of a prototype of a mathematics curriculum for pre-service teacher education. In this prototype is argued that the mathematics curriculum in pre-service teacher education should offer a mathematical matrix strong and flexible enough to enable them to manipulate and create the conditions for students to learn mathematics.

Students who fail to transform rote-learn arithmetic situations into simpler structures by compressing them into thinkable concepts can't relate them to conceptual learning thus stopping them to generate new knowledge in the sense that Gray & Tall (1994) called the *proceptual divide*. The episodes exemplified in this paper are taken from arithmetic tutoring sessions.

To teach the intuitive arithmetic studied in basic education, future teachers need a foundation or construction following a rational method. This distinction between rational and practical arithmetic does not arise from the nature of the subject, but from the method by which it is structured, founded on a deductive theory based on the boundary between arithmetic and logic by adding scientific rigour to the ability of reasoning that future teachers should have.

This view is in line with the ideas of Tall (2007), in developing his theory of mathematical growth by changing focus from process to concept, called *procept, and* seen as:

> An elementary procept is the amalgam of three components: a process which produces a mathematical object, and a symbol which is used to represent either process or object. (Gray & Tall, 1994, p.6)

According to this focus on the complexity of mathematical thought (Tall, 2002) we are trying to understand and find how pre-service teachers operationalize mathematical concepts in arithmetic accessing evidences of the *proceptual divide*.

The implementation of a new mathematical curriculum allows to study, characterize and test that curriculum in various perspectives and at the same time, by analysing the student answers in the various tasks like homework's, quizzes and exams, using for categories the SOLO (*Structure of Observed Learning Outcomes*) model from Biggs and Collis (1982) and Biggs and Tang (2007).

Associating the evolution of their mathematical thinking using Tall theories of *Advanced Mathematical Thinking* (Tall, 2002, 2007) and the concepts of *procept*, *proceptual thinking* and *proceptual divide* (Gray & Tall, 1994) as frameworks to perceive the way students conceptualize mathematical concepts.

BACKGROUND

The introduction of a new mathematics curriculum for Basic Education (pre-service kindergarten, primary and elementary teachers) is reflected in teacher education either by the particular definition of a new kind of student (more proactive ones) or by the need for new methods of teaching and learning of mathematics. This opportunity to change the mathematics curriculum for teachers occurs under a combination of factors: amendments to study plans in higher education; the growing concern with the teaching of mathematics in particular due to the poor results of international studies like PISA (OECD) and a significant change in the mathematics curriculum in elementary education.

Considering all these factors we designed a mathematics curriculum for Basic Education at an institution of higher education in Portugal who want to combine three levels of intervention: a solid mathematical foundation for all pre-service teachers; a comprehensive training for teaching mathematics connection among knowing and teaching mathematics; a didactics and pedagogical approach about what means to teach mathematics.

The curriculum has been developed taking into account the argument that the pre-service teachers need to have a better understanding of mathematical concepts because it helps them gain a greater understanding of the connections among different areas of mathematics and beyond, taking into account that this course is for generalist teachers.

TEACHING MATHEMATICS TO PRE-SERVICE TEACHERS

Issues related to mathematical preparation of future teachers has been investigated in view of training and teaching on education and not have as much importance as the subject of study for conceptual knowledge of these professionals. Studies on this topic have shown signs of concern, because this kind of mathematical knowledge is not present in many teachers (Veloso, 2004).

To Tall (1989) curriculum development should offer student contexts where they develop mathematical knowledge, leading to a significant growth of their mathematical reasoning. This arduous process of transitioning from a less formal mathematics to a more formalized understanding of mathematical processes and concepts, needs to be assessed by the teacher, both in terms of the complexity of thought highlighted and within the quality of learning.

The traditional concept of a mathematics curriculum structured gradually, starting with familiar elementary concepts and to a gradual complexity of structures has not worked for the simple reason that our brain does not function logically like a computer (Tall, 1989).

In addition, there are several conceptions that advocate that, on the one hand future mathematics teachers must have a broad base of didactics and pedagogical training and the knowledge of the mathematical content will be gained from the experience, on the other hand some argue that they should have a large mathematical training and that the pedagogical aspects are going to be acquired with the experience. In our view, both these dimensions should be balanced, and that should exist a compromise here.

The quality of learning

We consider in this study the quality of learning not only as the quantitative grade a student achieve when answering a question but also to the qualitative process of producing an answer using facts, concepts and skills to achieve the solution to that answer. But this is a complex issue because the quality of learning does not depend exclusively on the student, but there's other dimensions like the quality of the teaching itself, and other like is prior knowledge, motivation, self-regulated learning and so on.

In this paper we are experimenting the hierarchical SOLO model to identify only the process of producing an answer.

The SOLO model

Is the emphasis on the analysis of the quality of the responses from students that make the SOLO model interesting for the study. Throughout the development of the problems the focus is not on correct or incorrect answers, but in the structure (nature) of the responses, encoded in categories based on the SOLO levels enabling a more detailed description of the development of mathematical thought and the quality of their learning.

Table 1: Description of levels in the SOLO model relating them with the response indicators adapted from Biggs & Collis (1982) and from Ceia (2002)

	Mathematical thinking	Indicators
Extended abstract	Goes beyond the topic, make connections to other concepts and generalizations.	Theory, generalization, hypothesis, reflection. Maximum capacity, uses relevant data and interrelations. No felt need to give closed responses allowing possible alternatives.
Relational	Makes complex connections and synthesizes parts to the overall significance.	Compare, explain the causes, integrate, analyze, report, apply. High capacity, uses relevant data and interrelations. No inconsistencies within the subject, but closure is unique.
Multi-structural	Makes some connections but lack a unifying vision.	Enumerate, classify, describe, list, match, working with algorithms. Medium capacity, can isolate relevant data. Can achieve a different conclusion with the same data.
Uni-structural	Makes simple connections without identifying its importance.	Identify, memorize, perform simple procedures. Low capacity, only one relevant data. Jumps to conclusions on a single aspect.
Pre-structural	Provides information loose and disorganized, not related.	Can not relate. Minimal capacity, confused answer. Inconsistent responses.

This SOLO model becomes a tool which allows a framework which assists the implementation of an educational model based on the mathematical complexity of thought, in view of the quality of their learning and allows to avoid the emphasis on a single learning process.

The *proceptual divide*

The procedural approach of learning mathematics is rooted in positivistic frameworks where, through a set of predefined and outlined procedures, one gets an answer. This appears to be embedded on the idea that by doing sufficient numbers of similar (or even identical) exercises one gets the routines needed to learn mathematics. Although the short-term success of such approach is relatively obvious and educational policies adopt such measures, since they guarantee interesting statistical results (seen in the short term) - for example if a student preparing for an exam, repeating the exercise until exhaustion of previous examinations. In the long term it can be seen the flaws of such an approach, when that same student needs to relate content learned in previous years, so there is not a significant learning.

Gray & Tall (1994) use the concept of encapsulation of a process in a mental object, rooted in the work of Piaget to sustain cycles of building mental structures that in Piaget's theory are cycles of assimilation-accommodation, or reification in Sfard theory.

The use of symbols, however, have a double meaning, introducing some ambiguity between the procedure and the concept. The way students deal with this ambiguity seems to be the root of a quality learning of mathematics.

> We characterize proceptual thinking as the ability to manipulate the symbolism flexibly as process or concept, freely interchanging different symbolisms for the same object. It is proceptual thinking that gives great power through the flexible, ambiguous use of symbolism that represents the duality of process and concept using the same notation. (Gray & Tall, 1994, p.6)

This combination of procedural and conceptual thoughts is called *proceptual thinking*. When there is an inability to relate these two types of thinking making it is impossible the development of conceptual thinking.

The dichotomy between those who can not overcome the barrier of procedural thinking is defined by *proceptual divide*. This is one of the biggest barriers and one of the factors that has most contributed towards the failure of teaching and learning mathematics (Gray & Tall, 1994).

Teaching arithmetic's

The foundation of number theory focuses on, in their essence, in two schools, an Formalist represented by Peano and Hilbert among others, and another Logic represented by mathematicians such as Cantor and Russell. In our curriculum there was no concern of using any of the current theory of the integers or an axiomatic model. What is required in the course is just an introduction to elementary mathematics. It must be therefore an ordination of the theory in such terms that every proposition is a logical consequence of propositions previously demonstrated. The conceptualization thereby requires the establishment of ideas or primitive concepts defined by axioms.

"The fundamental idea in the development of powerful thinking in mathematics is the compression of knowledge into thinkable concepts." (Tall, 2007, p.150) This compression of knowledge enable students to relate ideas, concepts and *procepts* allowing them to go beyond rote-learning, that a number of studies show that fails in an unfamiliar context, like is showed in the second episode in this paper.

METHODOLOGICAL APPROACH

The examples in this paper show the process of inquiring and reasoning in action. These episodes are taken from one larger ongoing study in which we analyse and evaluate the mathematics curriculum of pre-service teachers and their learning. The aim of this study was to try to understand the mental construction of mathematical reasoning and, more specifically, to see how a student thinks mathematically taking into account the *proceptual divide* notion, hoping to observe it in action.

This specific experiment was designed based on a tutorial interview in which two students (let's call them Ana and Maria) came to clarify doubts on some issues that had been asked in the exam.

These students attended, and failed, *Mathematics I*, and requested these tutoring sessions to clarify doubts for the upcoming exam, so they had already attended classes, either theoretical either practical matters about these episodes, having made various of the exercises about the issue. These two examples were taken from the exam questions, and have been solved in the tutoring sessions.

In these episodes one of us acted as a teacher and as a researcher and two individual sessions of one hour per student were observed, totalling four hours of work. Both about one exercise of rational arithmetic.

One of the goals of this kind of exercises was to enhance not just the resolution of common arithmetic operations, but to establish a parallelism with aspects of a more formal mathematics, intending to develop mathematical reasoning.

By mathematical reasoning, or demonstration, we understand the combination or joining of two or more propositions to obtain new propositions by means of mathematical reasoning within a finite number of steps deduced from one or more propositions. The method worked is the complete induction (or method of recurrence) and to demonstrate that a given property is true for all integers is enough to demonstrate: (i) is true for 1, (ii) admitted as true for n, is also true for the successor of n (heredity).

Starting from the following proposition that is taken as an axiom:

> All property belonging to the integer 1 and the successor of an integer which enjoys this property belongs to all integers (principle of finite induction).

The operations studied were the addition, multiplication and exponentiation and are defined as:

1. Adding two integers - the operation that for each pair of integers *a* and *b* matches a given integer ($a + b$) according to the following conventions:

[A1] $a+1=suc\ a$

[A2] $a+ suc\ b = suc\ (a+b)$

2. Multiplication of two integers - the operation that for each pair of integers *a* and *b* matches a given integer ($a.b$), according to the following conventions:

[M1] $a\ .\ 1 = a$

[M2] $a\ .\ suc\ b = a\ .\ b + a$

3. Exponentiation - the operation that for each pair of integers *a* and *n* matches a given integer (a^n), according to the following conventions:

[E1] $a^1 = a$

[E2] $a^{suc\ n} = a^n\ .\ a$

Thus, it is necessary that the student in the following problem, realize these recurrence concepts to solve: *Calculate, by recurrence, using the respective axioms* $(2+2)\ .\ 2^1$

The categories of analysis of the question and subsequent answers are based on SOLO levels and their attributes and this exercise has been classified as possibly *relational* indicating an orchestration between facts and theories involved, their actions and goals. These kind of exercises were familiar to both students in the classroom environment although it was the first time they had covered this kind of procedure in mathematics.

To take a deeper analysis we used Tall theories covering an important aspect of the *proceptual divide* since the *procepts* here involved are ambiguous (operations can be viewed either in the field of elementary arithmetic or in the field of rational arithmetic).

EXPECTATIONS FROM THE PROCEPTUAL DIVIDE – TWO EPISODES

For Ana this issue had been solved naturally through elementary arithmetic and had not even noticed that, in the task she was asked to use the rational arithmetics, not understanding her low rating. The dialogue between Ana and Maria and myself was held in Portuguese.

Ana:	So, but the account is not right? Gave 8 ...
Teacher:	Yes, Ana, but read the statement again ...
Ana:	Yes .. and ...
Teacher:	We used the axioms?

Ana: What axioms? ... Ah! ... [reread the sentence, looking at the exam and then back at me with a satisfied air]

Teacher: ?

Ana: So I must do like in the classroom, with those axioms of operations, right?

Teacher: Right. [It is noted then a change in Ana, who quickly wrote on a sheet ... (2+2).2 (E1)... (2+ suc 1) . 2 (A1) ... (suc 2+1) . 2 (A2)... and so on until the right answer]

Teacher: Why you do not do that in the exam?

Ana: Well, I did not read the statement professor ... and it was so easy ...

After seeing her failure, Ana had no difficulty in solving the exercise. This is an example in which, looking at the question (not having read it entirely) and especially to the expression, used a familiar process of elementary arithmetic to solve (correctly) the question ... the issue is that she did not answer the question of the exam.

From the moment she really knows what to do, quickly solves the exercise using rational arithmetic's with axioms relating the properties studied by making a change in how she handles the mathematical objects.

In the exam she focused on the procedures looking for the expression in a disconnected way between number and operation, subsequently, by looking to the expression as a whole, she identified the respective axioms.

In this episode Ana easily surpassed the *proceptual divide* on this issue, rapidly changing their mathematical thinking from procedural to *proceptual* simply by reading again the question. When analysing the responses through the SOLO levels, Ana went from a response classified as a possible *uni-structural* level (in the reply to the exam) by jumping to fast for a conclusion, to a response in possible *relational* level (in the tutoring session) by generalizing within the given context using related aspects which corresponds to the level intended to the question. The next episode with Maria went up differently:

Maria: Professor I did not understand what it was is that I supposed to do in this exercise ...

Teacher: So, Maria?

Maria: asks to do math, but with what axioms?

Teacher: those who were worked in class, you don't remember?

...

Maria: but this was not just in lectures? I mean we had to decorate these axioms, it could come in a form ...

Teacher: why?

Maria:	We've done a few of these exercises, and always with an operation only, never with several, so one can not memorize everything ...
	[after almost an hour to explain the resolution and how it should identify the propositions]
Maria:	My head does not give much more, this is very complicated …
Teacher:	Tell me why?
Maria:	Only theory, when I teach kids, I will not teach so, teaching them only to do the math, none of these things successors and properties and axioms …
Teacher:	That is not so Maria ... (and we continue with a discussion of what it was to be a teacher ...)

In this example, Maria was so attached to the procedure that after two sessions she still didn't realize that she could not rote this kind of exercise. Even the very procedural knowledge (she couldn't understand the procedure) was deficient because she could even identify the numeric expressions as more complex *procepts*. In this case the *proceptual divide* is in line with the ideas of Gray and Tall when they state that:

> This lack of a developing proceptual structure becomes a major tragedy for the less able which we call the proceptual divide. We believe it to be a major contributory factor to widespread failure in mathematics. (Gray & Tall, 1994, p.18)

At the end of the sessions, Maria could solve some issues, but with only two integers and one operation, having been scheduled a third session, with some exercises proposed by me (which never came to happen until the next day of examination). The responses of Maria (on examination and tutoring session) were classified (according to the SOLO levels) as possibly *pre-structural* with great inconsistencies and later *uni-structural* by jumping to fast to conclusions, far from the outcome intended with the exercise.

FINAL REMARKS

In this paper we analysed two episodes in an attempt to expose the conceptualization of Gray and Tall on the *proceptual divide*. The work of Ana and Maria (like all his colleagues) is to overcome the barrier of elementary arithmetic and think mathematically about a set of properties that are common to the operations studied in any circumstance, through the generalization that is possible in the rational arithmetic and this separation was identified by their different characteristics.

The use of the SOLO levels to characterize the questions and answers has the limitation of identifying an image at the moment it is not possible, and in isolation, assess the students' mathematical thinking, hence the necessity to use other methods such as inquiry, to obtain a better understanding of the phenomena under study.

Ana's success depended more on concentration and ability to read the sentence, since after a few questions (directed to read of the sentence and not to the mathematical content), managed to solve that problem, and others - and subsequently had a positive rating the exam by solving a similar issue using other properties of operations so it's unclear that she crossed the *proceptual divide* by means of the intervention (which was not the aim of this paper) or that she already is using *propectual thinking* and the problem with the exercise was only due to misreading, Maria failed again in the exam, but in a similar issue she solved partially (only the addition, which was detached), revealing still be using only procedural thinking, having failed to overcome the barrier of *proceptual divide*.

Both examples are used to identify differences in the mathematical thinking related to the same exercise and require further and clearer evidences of the *proceptual divide* that could be an important mechanism for mathematics teachers and researchers in mathematics education.

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