SPECIALIZED AND HORIZON CONTENT KNOWLEDGE – DISCUSSING PROSPECTIVE TEACHERS KNOWLEDGE ON POLYGONS

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The specialized and horizon content knowledge (SCK and HCK), two of the MKT (Mathematical Knowledge for Teaching) sub-domains, are the focus of this paper. We focus our attention on these two sub-domains of teachers' knowledge due to the delimitations problems concerning the content of SCK and the need to discuss the nature of HCK with greater precision. From a case study focusing on one prospective secondary teacher we present some descriptors concerning the concept of polygon, and discuss the desirable content of teachers' specialized knowledge and its delimitation problems.

Key-words: Mathematical knowledge for teaching (MKT), specialized content knowledge, horizon content knowledge, polygons, prospective secondary teachers.

INTRODUCTION

The teaching-learning process is influenced by a large set of factors (e.g., beliefs, goals, class/school size). One of the most influential factors concerns teachers' knowledge (e.g., Nye, Konstantopoulos & Hedges, 2004). Such knowledge influences, inter alia, focus, the pursued goals, and, consequently, what and how students are intended to learn, as well as the view of mathematics they are intended to experience and acquire. Thus, it is of fundamental importance to call attention to teachers' knowledge at all moments, being one of the more crucial initial teachers training (Llinares & Krainer, 2006) – as it's the moment of access to the work of teaching from the teacher point of view and not from a student point of view. Such call of attention can begin by accessing and exploring prospective teachers' knowledge while discussing and reflecting on a mathematical topic (or some "hypothetical" students' comments) in a questionnaire or while planning or implementing teaching sessions.

Although there is almost unanimous acceptance of the influence of teachers' knowledge on students' learning, what is comprised in such knowledge, its type, content and specificities (or not) can be understood differently under different perspectives (e.g. Rowland, Huckstep & Thwaites, 2005; Davis & Simmt, 2006).

From amongst the most recent and influential conceptualizations on teachers' knowledge in the course of our recent work, we opted for the Mathematical Knowledge for Teaching (MKT) model (Ball, Thames & Phelps, 2008). This option was made taking in consideration the very specific orientation to the teacher's mathematical knowledge of MKT, placing emphasis on the mathematical reasoning involved in the work of teaching (understood as more than correcting mistakes and evaluating students' answers). Such a choice was also connected to our perception of the sub-domains of the MKT as a relevant starting point for designing tasks for the mathematical preparation of teachers, and for doing research on what inputs to teachers training show effects on students and practices. Intending to contribute to an advance on the understanding of the MKT and on the content of its sub-domains, we began research focusing on discussion of the content of some of the sub-domains. That broader research includes work focusing on teachers practices from kindergarten till Higher Education and is leading us to question some particular aspects of the MKT conceptualization concerning the similarities and differences in the content of some of the sub-domains (Carrillo, Climent, Contreras & Muñoz-Catalán, 2012). In this paper we focus on data from a questionnaire applied to seven prospective secondary teachers' focusing on polygons and we will discuss, particularly, the specialized and horizon content knowledge revealed by Marta, one of these prospective teachers. Such a discussion also aims to call attention to the need of a deeper specification of the content and nature of these sub-domains.

SPECIALIZED CONTENT KNOWLEDGE (SCK) AND HORIZON CONTENT KNOWLEDGE (HCK)

One of the core aspects that the MKT brings to light concerns the specificities of teacher's knowledge for teaching, when compared with the knowledge involved in other professions in which the same topic (mathematics) is, also, used.

Having the genesis on Shulman's (1986) work, the MKT conceptualization considers the two domains of teachers' subject matter knowledge (SMK) and pedagogical content knowledge both divided in three sub-domains (see Ball et al., 2008). Our focus here concerns only the sub-domains included in SMK. The MKT conceptualization considers that Common Content Knowledge (CCK), the knowledge that allows one to know how to do mathematics for oneself, although necessary, is neither sufficient nor specific for the teaching of math. Thus, included in the content domain is a specialized Mathematical Knowledge for Teaching (SCK), which differs from the knowledge needed in settings other than teaching and also another kind of knowledge, related with how things can be perceived interconnected along schooling (HCK) – but not related with the content being approached at the moment.

The SCK is perceived as *the mathematical knowledge and skills unique to teaching* and thus, it only makes sense to be part of teachers' knowledge and not necessarily part of the knowledge needed outside teaching. Such knowledge implies *the use of decompressed mathematical knowledge* (Ball et al., 2008), in such a way that teachers are in a position that allows them to understand and explain (with and for understanding), amongst others things, the mathematical reasons embedded in students' errors or alternative reasoning; generate or reformulate examples that reveals key mathematical ideas; or identify the critical characteristics of a given definition or when elaborating one. SCK is, in this sense, seen as the knowledge required by the teacher who genuinely wishes their students to understand what they do, and not merely mechanically run through a set of given procedures – this is not limited to procedures but has a broader meaning that includes the necessary concepts and how they interrelate and influence each other (Ribeiro & Carrillo, 2011).

On the other hand, HCK can be understood as an awareness of how the current mathematical topic fits into the overall scheme of the students' mathematical education, how the various topics relate to the others, and the way in which the learning of a particular topic may relate with others as one moves up the school. (We have to remark that HCK does not concern the knowledge immediately related with the content being approached at the moment (Jakobsen, Thames & Ribeiro, 2012)). It is thus a kind of knowledge that is neither common nor specialized and that is not about curriculum progression. It is knowledge about having a sense of the larger mathematical environment of the discipline (Ball & Bass, 2009), allowing one to perceive different types of connections between and amongst different topics or even in the space of one topic at different school levels or in one level (e.g., Fernández, Figueiras, Deulofeu & Martínez, 2011). Such knowledge (even in these complementary perspectives) can enable teachers to make judgements on what is mathematically important and worthwhile to pursue – not being part of the curriculum they are supposed to teach – at a particular moment/school level. Such knowledge can also be perceived as an advanced mathematical knowledge from an elementary mathematical stand point (Montes, Aguilar, Carrillo & Muñoz-Catalán, 2012), meaning that advanced mathematics for teachers need to be demonstratively related to the teaching that takes place in school.

In our research group, and what concerns specifically the work focusing specifically on teachers' knowledge, we went across several stages: firstly, our group aimed to identify, from observed classroom practice, the knowledge teachers show evidence of deploying at every specific moment (e.g., Ribeiro & Carrillo, 2011; Sosa & Carrillo, 2010); the knowledge prospective teachers evidenced when answering a questionnaire/solving a task on fractions or polygons (e.g., Ribeiro & Jakobsen, 2012; Carreño & Climent, 2010) or even when interpreting and giving sense to students nonstandard productions (e.g., Jakobsen & Ribeiro, 2012). Such an approach leads us to a complementary focus of attention, in which we discuss and reflect upon the content of the MKT sub-domains and the conceptualization itself (e.g., Carrillo et al., 2012; Montes et al., 2012).

Although there is a certain consensus that there are no well-defined borders around MKT domains, such undefined borders make it difficult to differentiate what is included in each one of the domains (e.g., when analyzing a classroom practice, a questionnaire or an interview).

Specifically, if we are looking at what could be considered for inclusion in the SCK, we face some problems in achieving such a unique identification because it might be impossible to say, for certain, if a certain aspect of knowledge is specifically related with SCK or could be included on the CCK or even, according to with some perspectives (as connections), included in the HCK. These difficulties may be related to the fact that all these sub-domains have a straight relationship with the SCK (it influences how teachers perceive all the teaching-learning process and, thus, all the others domains) and thus with the specificities of teachers' knowledge when compared with other professionals who use mathematics as a tool. Such difficulties are perceived as one of the limitations of MKT and, if the aim is to call attention to the specificities of teachers' knowledge, it should be possible (at all times) to distinguish the nature (and content) of all its sub-domains.

These limitations of the MKT lead to a possible (re)conceptualization of teachers' knowledge (e.g., Carrillo et al., 2012), having as a starting point the problematic of the specificities of teachers' knowledge and if such specificities are better perceived as integrated in the previously mentioned sub-domains and not only in one (or some) of them. In such a process of deepening and clarifying the nature and content of the mathematical knowledge of the mathematics teachers, we felt the need to characterize each one of the involved MKT sub-domains, which lead to an enlargement and more refined delimitation of each of the components of teachers' knowledge. A discussion of its similarities and differences is central in this process.

METHOD AND FINDINGS

In this paper we discuss a prospective teacher's (Marta)'s SCK as revealed in a questionnaire about the concept of polygon as object of teaching and learning.

Marta is 22 years old secondary prospective teacher and she answered the questionnaire in her last year of initial training. Her academic results in the mathematics courses were, in general, low, but she revealed wiling of reflection and discussion on her own knowledge which was one of the reasons for being part of the study. She had also some experience in teaching, as she worked already as teacher in a private school (without the teacher degree), and thus she had some contact with pupils and their reasoning and argumentations.

Aimed at contributing to a deeper knowledge and understanding of teachers' knowledge, we will discuss and reflect on the nature and content of the SCK on the MKT conceptualization. Thus, we will not focus on describing Marta's MKT but we will focus specifically in presenting and discussing some difficulties in determining the boarders of the SCK, in concrete related with HCK. To analyze the questionnaire, an instrument has been elaborated with some descriptors of desirable mathematical knowledge for teaching the topic of polygons. Some of these descriptors were elaborated from the literature review (research on students' and teachers' knowledge and difficulties on the topic – e.g., Tall & Vinner, 1981; Fischbein, 1993; Fujita & Jones, 2007; Battista, 2007) and others emerged during the analysis process.

When elaborating each question we thus needed to identify the sub-domain such a question was addressing. One of the possible ways to overcome such a difficulty was to equate and question ourselves (and, explicitly, other experts) as to whether a certain specific aspect of knowledge on polygons would, or would not, be shared with other professions – such as mathematicians. Although we are not absolutely certain of the limit of the mathematical knowledge of a mathematician on this topic, it was assumed that a mathematician could know a definition of a polygon without linking it to the basic necessary concepts (sub-concepts) that are fundamental in order to be able to elaborate the notion of such a geometric being with and for understanding. We considered that such knowledge related to linking the sub-concepts to elaborate a geometric notion is necessary and somehow exclusive for developing the tasks of teaching (e.g., choosing and developing useable definitions; selecting representations for particular purposes). One other option had to do with the fact of considering (after some discussions with persons with some mathematical training) that in any other profession would be necessary (involved) to note the critical characteristics of a polygon, being that such knowledge is essential for teaching mathematics while elaborating/exploring a definition of polygon. Such centrality has to do with, amongst other things, the fact that these critical features correspond to the ones students' are supposed to note while differentiating polygons from non-polygons.

From the previous ideas, assuming the sub-concepts and critical characteristics as core elements in a definition of a geometrical concept, we have elaborated some indicators aimed at describing prospective secondary teachers' SCK. We have selected three of them to discuss here (our choice is due to the fact that these are the ones more present in the prospective teachers' answers and, on the other side, because they allow for illustration of the difficulties in defining SCK borders).

> ***SCK1:*** *Know to identify the concepts and sub-concepts involved in a geometrical concept.*

Understanding a sub-concept as a specific concept that allows the elaboration of a more general concept, it is assumed that a teacher must have knowledge that allows for its identification. Such identification allows for the elaboration of a deeper sense and understanding of the concept itself, with an ampler sense and not associated to prototypes, and thus also allowing for detection of students' conceptual errors and misconceptions, being useful while elaborating a verbal definition of a geometrical object. While discussing some aspects related with the sub-concepts, Blanco & Contreras (2002, p. 112) mentioned that, in order to represent (draw) the orthocenter of an obtuse triangle it is necessary to know the concept of orthocenter and high of a triangle; but in order to be able to represent this last concept of high of the triangle, it is essential to have an understanding of the sub-concepts of perpendicularity from a point, of a line perpendicular to each other and of vertex opposite to a side of a triangle. Such knowledge is essentially mathematical knowledge but specific to the task of teaching, which justifies its inclusion in SCK. On the other hand, identifying the sub-concepts implied in a certain geometrical concept implies the knowledge of a net of relationships between concepts (which is also perceived as HCK in Fernández et al., (2011)) and differentiating them between more or less fundamental (basic) concepts (in the sense of Ma (1999)) and others in which such sub-concepts form a part. This idea of mathematical structure is one of the reasons that makes us doubt if such knowledge, which was initially, in an obvious way, included in SCK, should be included in the HCK – as defined also by Ball and colleagues.

In one of the questions, several different (pseudo)definitions of polygon were given and the prospective teacher had to comment on them and discuss the sub-concepts involved. (There was a clarification in the question of what meant a sub-concept – necessary concepts know in order to be able to understand the concept of polygon.)

Three such (pseudo)definitions were: (i) *it is a geometrical shape with sides and different angles. They can be concave, convex, regular and irregular*; (ii) *a polygon is a geometrical figure with equal sides and angles*; (iii) *it is the interior of a geometrical closed figure made by the lines that join three or more dots and the dots should not cross*.

When analyzing the given definitions, Marta points as sub-concepts only those that are involved explicitly in a given definition. When commenting on the first two (pseudo)definitions she considers as sub-concepts: geometrical figure; geometrical body; angle; classification of angles; measurement of a line (as an element of a polygon). But in the third she introduces a nuance considering "interior region" as a differentiating sub-concept associated to a critical characteristic to differentiate between what is and is not a polygon (such an idea is reinforced when she proposes a four-sided polygon, with the sides crossed, as a way of identifying the error in pseudo definition, as it would define two regions (iii)).

> ***SCK2****: Know to identify the critical features (characteristics) of a definition or given notion of a geometrical object*

Perceiving as critical features (characteristics) the ones that are necessary and sufficient to define a concept, we assume this corresponds to the knowledge the teachers must possess to distinguish the general characteristics from the specific or accessory ones. As an example, from Marta's answers in the first given (pseudo)definition (mentioned previously in (i)), when discussing the misunderstood features (they can be concave, convex regular and irregular) she mentions that, "*here the student is confusing classification by its angles and by its sides with the* *definition*". In such, she reveals that in the definition of polygon there is no need explicitly to classify and she thus describes such a characteristic as accessory rather than critical. Putting together this evidenced knowledge and the previously mentioned in SCK1 concerning Marta's knowledge (a polygon is closed and has only one interior region), it lead us to wonder if knowing these defining characteristics of the concept of polygon can be considered as common content knowledge, not being exclusive from the teachers' knowledge space.

SCK3: *Know to identify the critical characteristics when elaborating a definition of a geometrical object.*

The elaboration of a geometric definition is influenced by the choice of the fundamental concepts to focus on and the critical characteristics that allow differentiating two different objects. In Marta's case, when asked to give her own definition, she includes as sub-concepts and critical characteristics some of the ones she discussed previously in the students' (pseudo)definition. For Marta, a polygon is: "*a geometrical flat closed figure with a polygonal region and its border is the union of three or more non-collinear dots".*

Although Marta considers a set of critical characteristics that may give us an idea that she has a specific knowledge that can be associated with the SCK on what a polygon is (flat and closed figure; interior of a polygonal line; with the minimum of three points), she mainly reveals herself to have knowledge related with the structure of a mathematical definition. Such a nuance is essential when analyzing teachers' SCK and it can be better discussed by analyzing Marta's examples, and non-examples of polygons in light of her given definition[1]. When asked to represent polygons and non-polygons, in light of the definitions she presents, we could expect her to draw convex and concave or even complex polygons (with sides crossed) or simply closed flat forms as a circumference, but she draws the following representations:

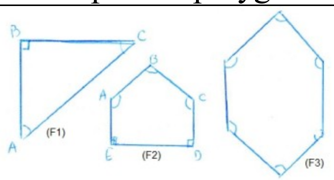
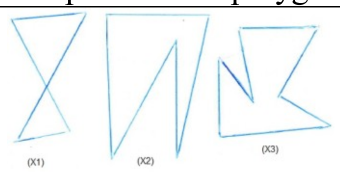
Examples of polygons	Examples of non-polygons
	

Figure 1: Examples and non-examples of polygons drawn by Marta

Putting the puzzle together (both the critical characteristics she presents and the representations she draws), it can be seen that the concept image she has contributes more information to her definition of polygons. Such an image leaves aside the critical characteristics of a polygon (in her own notion/knowledge of it), which lead

[1] These examples, and non-examples also, are obviously related to the example space she possesses and we, as teacher trainers, are supposed to allow teachers and prospective teachers to develop.

to a deficient/imprecise and incoherent definition with the mental image she has of the geometrical object.[2] Such discussion lead us, again, to equate if, when elaborating a definition, the critical characteristics are not more closely related with knowing what a mathematical definition is than with an SCK.

CONCLUSIONS AND FUTURE WORK

By analyzing a prospective secondary teacher's answers to one questionnaire, some questions emerged concerning both their revealed knowledge and the nature and content of teachers' SCK and HCK. Some of these questions concern the prospective teachers' knowledge on the role of the sub-concepts in the definition (and thus, what a definition is and the different possible approaches in defining a certain geometrical object). Examples of such a questions are: is the identification of the sub-concepts and the acknowledgement of the critical characteristics associated to polygons part of SCK or is it part of CCK (in the sense that any person with some mathematical training is supposed to have a knowledge on these sub-concepts and its role in the definition)?; Is the identification of the sub-concepts an SCK or a specific knowledge related with connections and thus in the space of HCK (in the sense of the HCK conceptualization presented by Fernández et al. (2011))?; Is the fact of considering the critical characteristics when elaborating a definition (of a polygon) related to the nature of mathematics itself?

The previous questions are part of the examples that lead us to discuss and reflect on the limitations (and difficulties) in differentiating some aspects of the MKT sub-domains in terms of its content and nature and in particular on which aspects of the mathematical knowledge are specific for teachers to teach mathematics with and for understanding. Aligned with this problem, our research group has started working specifically on the space of the knowledge a teacher has/must have in order to perform their work of teaching, leaving aside a discussion on whether such knowledge is shared with other professions or not. Carrillo et al. (2012) present a theoretical discussion of such an approach and its impact on the conceptualization of teachers' knowledge. Our previously proposed questions are one of the kinds of questions at the core of the motives that lead to such work and we thus now perceive as necessary not only a change in our descriptors (allowing for leaving aside the tasks of teaching – as all the discussion will be in the field of teaching – and allowing a deeper description and discussion of the knowledge) but also allowing a deeper discussion on the nature of the knowledge that each one addresses. Further research is needed in order to clarify the content of each of the domains. Such identification and discussion would allow for the creation of a body of knowledge

[2] Although the differences between conceptual image and conceptual definition in the same object have been previously discussed (e.g., Tall & Vinner, 1981) we consider that in order to be able to reflect on the critical characteristics it is essential to have an awareness of the characteristics that limit the concept definition, leading to a harmonization of image and definition.

allowing a deeper and more fruitful focus of attention on the more problematic and promising aspects of teachers' training. It would allow us to focus our attention where it is most needed, as well as reinforce the need for a specialized knowledge for the teaching of mathematics.

Acknowledgements:

This paper has been partially supported by the Portuguese Foundation for Science and Technology (FCT).

This paper forms part of the research project "*Mathematics knowledge for teaching in respect of problem solving and reasoning*" (EDU2009-09789EDUC), funded by the Ministry of Science and Innovation in Spain.

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