# MTSK: FROM COMMON AND HORIZON KNOWLEDGE TO KNOWLEDGE OF TOPICS AND STRUCTURES

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***Abstract***

*This paper focuses on the mathematical knowledge a teacher needs to be able to teach. We give particular consideration to the principles underlying the subdomains making up the model, MTSK (presented in Carrillo, Climent, Contreras and Muñoz-Catalán (2012)), building on the category of Subject Matter Knowledge (Shulman, 1986). We define and analyse three subdomains: Knowledge of Topics (KOT), Knowledge of the Structure of Mathematics (KSM), and Knowledge about Mathematics (KAM). We discuss the defining features of these categories, contrasting them with the model of MKT developed by Ball et al (2008), and using examples from our own experience as researchers in the area.*

***Keywords:*** MTSK, MKT, knowledge of topics, knowledge of the structure of mathematics, knowledge about mathematics.

## *FROM MKT TO MTSK*

Ever since Shulman's (1986) seminal work, setting out the knowledge teachers bring into play in the exercise of their profession, a separation has been recognised between Subject Matter Knowledge and Pedagogical Content Knowledge, with seven different categories making up these two main domains. In their construct of Mathematical Knowledge for Teaching (MKT), Ball, Thames & Phelps (2008), present a classification of mathematical knowledge, following Shulman (1986), and introducing six different subdomains, *Common Content Knowledge* (CCK), *Specialised Content Knowledge* (SCK), *Horizon Content Knowledge* (HCK), making

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up Subject Matter Knowledge and *Knowledge of Content and Students* (KCS), *Knowledge of Content and Teaching* (KCT) and *Curricular Knowledge*, constituting Pedagogical Content Knowledge. We consider that SCK is particularly relevant as it is considered an area of knowledge exclusive to the profession of mathematics teaching. Despite representing a significant advance in our understanding of mathematics teachers' knowledge, the definition of this subdomain overlaps with others (Flores, E., Escudero, D.I. & Carrillo, J. (2012)). Such problems of demarcation between subdomains led to the development of the MTSK framework (Carrillo et al. (2012)).

In the following analysis we try to differentiate the specific knowledge that is held by teachers about a mathematical item from the perspective of a pupil (derivatives, for example) from the knowledge of that same item as an element of common knowledge.

Ball et al. (ibid) emphasise the mathematical demands entailed in teaching, which they exemplify with a subtraction computation. The example gives the correct answer to the subtraction 307-168 via the so-called 'borrowing' algorithm, then considers various typical wrong answers by pupils, to understand the cause of which requires special mathematical reasoning, and finally proposes other non-standard approaches which are often unfamiliar to the teacher. They state that anybody who knows how to solve the calculation can identify when a pupil's answer is incorrect, but that "skilful teaching requires being able to size up the source of a mathematical error" (ibid. p. 396). This kind of teacher knowledge, they add, is complemented by an ability to do such an analysis efficiently and fluently, and to see beyond the errors to the particular problems facing the pupils.

We would agree that identifying a wrong answer of this kind should be considered commonly held basic knowledge, and that it logically forms part of the teacher's knowledge by virtue of the demands of the work of teaching (Common Content Knowledge). However, in the process of analysing the error the teacher brings into play two different types of knowledge. First, in the case of the non-standard approaches, the teacher needs to ask him or herself, "What is going on mathematically in each case?" (ibid p. 397). The answer to this question implies the mobilisation of intrinsically mathematical knowledge about the significance and implication of each step in the process of subtraction.

Second, in the case of a student arriving at the answer 261, the teacher has to consider "what line of thinking would produce this error" (ibid p. 396), and reflect on the nature of the misunderstanding that gave rise to the mistake. In this case, the knowledge brought to bear is not only mathematical, as the item under consideration ceases to be mathematics itself and becomes the cognitive processes called upon when a pupil tackles a mathematics task.

In this paper our interest is in gaining a better understanding of the mathematical knowledge (in the sense of Subject Matter Knowledge (Shulman, 1986)) required by teachers in their day to day practice. We approach this interest from the model of Mathematics teachers' specialised knowledge (MTSK) which we present in Carrillo, Climent, Contreras & Muñoz-Catalán (2012, in this volume). MTSK offers a new perspective on the knowledge required in mathematics education, and whilst respecting Shulman's (1986) original division between *Subject Matter Knowledge* and *Pedagogical Content Knowledge*, it brings two fundamental aspects to the fore. First, it adopts the term "specialised" from the model of MKT by Ball et al (2008), but applies it to the whole of the new model. That is, instead of talking about specialised content knowledge, whereby the notion of 'specialised' is applied to content knowledge, the new model concerns the specialised nature of mathematics teachers' knowledge. Secondly, it shifts the focus of study onto the object of the teacher's reflection. Hence, in 'subject matter knowledge' we propose the subdomains *Knowledge of Topics* (KOT), *Knowledge of the Structure of Mathematics* (KSM), and *Knowledge about Mathematics* (KAM). In the category *Pedagogical Content knowledge* we include the following subdomains: *Knowledge of Mathematics Teaching* (KMT), *Knowledge of Features of Learning Mathematics* (KFLM) and *Knowledge of Mathematics Learning Standards* (KMLS). For a full description of all these subdomains, see Carrillo et al., 2012, in this volume. Our interest in this paper is to describe the principle features of the subdomains within the first group, giving examples from each.

## *KNOWLEDGE OF TOPICS*

In MTSK (Carrillo et al., 2012, in this volume), the subdomain *Knowledge of Topics* (KOT) represents a new way of viewing the subdomains *Common Content Knowledge* and *Specialized Content Knowledge* (SCK) from the model of MKT.

Ball et al. (2008) define *Common Content Knowledge (CCK)* as "the mathematical knowledge and skill used in settings other than teaching" (p. 399). We can equate this to the mathematics that can be found in mathematics (text) books (at any level). However, although this knowledge might be shared with other professions, we would argue that the teacher possesses a greater range and depth with regard to this knowledge by virtue of the simple fact that mathematics is the lifeblood of their work. This notion, as suggested above, inspires the idea that “specialised” spreads across the full range of domains in the model of MTSK (Carrillo et al., 2012, in this volume)

With this in mind, we can indicate other aspects of mathematical knowledge which should be included in this subdomain (KOT). To start with, there is the advanced knowledge needed to understand any particular topic. For example, if a teacher is explaining surface integrals at higher secondary level, they will need to know the concept of area, the density of rationals in the set of real numbers, and topological theory, all of which have in common the fact that without them, the topic cannot be understood and which are therefore required within the subdomain. By the same reasoning we would include non-curricular mathematical knowledge such as unconventional procedures for doing mental arithmetic.

Secondly, we also include knowledge about the different meanings a topic might have, as is the case with fractions (Llinares and Sánchez, 1997). A teacher who formulates different problems which can be solved via the same fraction but with different meanings demonstrates considerable reflection on the concept of fractions and their elements, which goes beyond mere problem solving and doing calculations with fractions. We also consider here phenomenological aspects associated with the knowledge of a mathematical item (Freudenthal, 1983; Rico 1997). In the model of MKT, all the above would be included within the subdomain *specialized content knowledge* (SCK). However, bearing in mind that the notion of specialised pertains to the full MTSK model, we consider that this kind of knowledge, being “pure subject matter knowledge” (that is, unalloyed with knowledge of students and pedagogy), belongs within KOT. Put simply, we consider KOT a subdomain containing advanced mathematical knowledge exclusive to the work of teaching.

To recapitulate, the defining features of KOT comprise advanced knowledge of school mathematics topics, along with knowledge of any different meanings involved, and corresponding phenomenological aspects.

## *KNOWLEDGE OF THE STRUCTURE OF MATHEMATICS*

In this section, we focus on the second subdomain within ‘subject matter knowledge’ in the model of MTSK (Carrillo et al., 2012, in this volume) denominated *Knowledge of the Structure of Mathematics* (KSM).

This subdomain emerged as a result of reflecting on Horizon Content Knowledge (HCK) in the MKT model, which is defined as “an awareness of how mathematical topics are related over the span of mathematics included in the curriculum” (Ball et al. 2008, p. 403). Initially, the authors were unsure whether this category belonged in subject matter knowledge or was spread across several categories. Later, Ball and Bass (2009) proposed subdividing HCK into three related dimensions: HCK - topics (HCK (T)) which concerns connections both within the field of mathematics and with other disciplines; HCK - practice (HCK (P)) dealing with how mathematics is constructed; and HCK - values (HCK (V)) specifying the main values when doing mathematics.

The consideration of how mathematics is interconnected internally is a significant component of KSM as it enables us to understand how teachers construct their mathematical knowledge. Likewise, knowledge of connections with other disciplines enables the teacher to devise mathematics problems drawing on other areas of knowledge. However, for the moment we will leave external connections to one side.

Martínez et al. (2011) consider that the *connections* teachers make between different areas of content (concepts or procedures), whether at the planning stage, during the execution of the lesson, or after the class, can be grouped into three types:

- Intraconceptual connections: connections between different ideas associated with a particular mathematical concept, constituting the essence of mathematics.
- Interconceptual connections: connections to different mathematical concepts.
- Temporal connections: connections between mathematical concepts at different stages of the curriculum, that is between what has been studied and what will be studied.

These connections derive from studies carried out by Fernández et al (2011), in which HCK is considered “not as another subdomain of MKT, but as a mathematical knowledge that actually shapes the MKT from a continuous mathematical education point of view” (p. 2646).

With regard to intraconceptual connections, we have become aware through successive analyses that the connections made are often so closely related within the same topic that we have been led to reconsider the appropriateness of including these within *Knowledge of the Structure of Mathematics* (KSM). The characteristics of such connections are closely linked to the knowledge of the meaning of a specific item, for which reason we consider them as pertaining to the subdomain *Knowledge of Topics* (KOT) presented above.

Interconceptual connections relate one mathematical topic to another or others. These connections are of a different order to intraconceptual connections as they underline that the mathematics teacher is establishing links between different mathematical items, in terms of both concepts and properties between them. The knowledge a teacher demonstrates in developing new mathematical items from existing knowledge is also included.

We include temporal connections in KSM, that is, connections to retrospective and prospective content relative to the current item of study. We can see a convergence here of the idea of *elementary mathematics from an advanced point of view, and advanced mathematics from an elementary point of view*. "These two notions enable two-way connections to be made – prospectively between elementary material at any particular level and its corresponding advanced treatment at later stages, and retrospectively between advanced material and its more basic treatment at lower levels," (Carrillo et al., 2012, in this volume). In this context, and complementing the idea of connections, we include the concept of increasing complexity (and conversely, simplification). For example, the procedure for classifying two-dimensional shapes is the same at secondary level as it is at infant level, but if teachers at each level were to tackle this area from a perspective of continuity between the two educational levels, each would take a very different mathematical view. The secondary teacher, for example, could deal with the procedure from a complex perspective, suggesting classificatory systems based on two or more criteria at a time or which involved the use of inclusive groups. The infant teacher, for their part, might suggest classificatory strategies that use visual and manipulative cues one after the other to make different groups; they might even use a simplified strategy, such as *identifying* attributes and qualities, or *comparing* shapes (in order to help the pupils at this age to discard their tendency to view the world as a homogeneous whole), which we believe lie at the mathematical heart of the classificatory procedure. In other words, the cline from simple to complex is not cognitive but mathematical; the teacher's task is to scrutinize the area of study so as to identify those contents which are close or connected and make up a mathematical framework.

Being able to superimpose a more advanced approach to a learning item onto a more basic approach to that item, or put another way, conceptualising an item from both a basic perspective and at the same time a more advanced perspective than that required by the curriculum at that point is one of the key specialised skills a teacher needs to develop.

As mentioned at the start of this section, KSM belongs to the category *Subject Matter Knowledge*, as does KOT. However, while the subdomain KOT concerns in-depth knowledge of a specific topic (getting to its mathematical heart), KSM concerns knowledge of how topics are interrelated, and this kind of knowledge is based more on a global understanding of the mathematical structure connected to a concept. In the case of KSM, to make an analogy with building a house, we are talking about knowing how to put one brick on top of another, which brick to put and why. In the case of KOT, it is a question of taking a step back to see the girders and joists which make up the framework supporting the house.

### *KNOWLEDGE ABOUT MATHEMATICS (KAM)*

One of the dimensions of the HCK classification devised by Ball and Bass (2009) is knowledge of the ways of knowing and creating or producing in mathematics (syntactic knowledge). This includes aspects of mathematical communication, reasoning and checking, providing and applying definitions, making connections (between concepts, properties and so on), using correspondences and equivalences, deploying representations, arguing, generalising and exploring. Our perspective is that this categorization corresponds to *Knowledge about Mathematics*, that is to say, conceptual knowledge about the rules of syntax themselves, and procedural knowledge about how to do mathematics, in addition to knowledge about the history of the discipline and its relation to other fields (Ball and McDiarmird, 1990). In this respect, we have included *Knowledge about Mathematics* (KAM) as the third subdomain making up *Subject Matter Knowledge*.

Again, setting to one side connections with other areas of knowledge as mentioned above, this kind of knowledge is what in large part can be considered “mathematical logic”. We understand that this knowledge, once again, is not exclusive to teaching, but is likely shared by anybody who has wondered about how mathematics is constructed. However, it is specialised in the sense that a teacher should know, for example, when a result with a double implication is fully or only partially demonstrated, and it is knowledge of this kind that enables the teacher to decide this. In short, then, *Knowledge about Mathematics* (KAM) concerns reflecting upon ways of doing mathematics.

A group of primary teachers in a collaborative professional development project pondered the question of whether “the addition of a multiple of 2 and a multiple of 10 results in a multiple of 10” (Muñoz-Catalán, 2012). One of the teachers replied that it would need a general demonstration, as sometimes this was the case, but other times it was not. This is an example of knowledge about ways of demonstrating mathematics, which for this teacher seems to consist in checking whether an affirmation that she knows is not always true, is always true. This knowledge represents an idea of demonstration that is not valid in mathematics, but which we can say is *her* way of understanding what demonstrations are. However, this knowledge is not used in isolation in this case, but in conjunction with the knowledge the teacher has of natural numbers and their properties of divisibility, a fact which underlines how KAM is a subdomain which occurs across domains when the classroom treatment of any mathematical topic is analysed.

## *CONCLUSION*

In this paper we give a detailed characterisation of the subdomains within the MTSK framework (Carrillo et al., 2012, in this volume), corresponding to the category of *Subject Matter Knowledge* proposed by Shulman (1986). We believe that the model of MTSK represents an advance in the process of categorising mathematics teachers' knowledge, which was initiated by Shulman and developed by Ball and the research group at the University of Michigan. It is the result of a considered analysis of the distinct dimensions of knowledge held by mathematics teachers, with the result that the subdomains meet classificatory criteria in terms of the object or aspect of the particular teaching item under consideration. The three subdomains considered here correspond to different objects within mathematics itself: *Knowledge of Topics* (KOT) refers to the knowledge of content itself; *Knowledge of the Structure of Mathematics* (KSM) concerns knowledge of the structure of the content; and *Knowledge about Mathematics* (KAM) is knowledge about how mathematics is constructed.

Regarding the dimension of HCK relating to values, we wonder whether the central values inherent to mathematics are really epistemic in nature. Reconsidering the exemplification for this subdomain given by Ball and Bass (2009) which they characterise as 'core mathematical values and sensibilities' we find elements such as 'precision, care with mathematical language, consistency, parsimony, coherence and connections'. We believe that, in the absence of a better characterisation, the items included in this subdomain are conceptions and beliefs about mathematics – such as mathematical attitudes – and towards mathematics, and cannot thus be considered knowledge. Nevertheless, after reviewing numerous works studying knowledge for teaching, we think that extending the framework to include such considerations can only bring a finer-grained sensitivity to the analysis, providing a broader and deeper snapshot of the teacher under study, and consequently of their knowledge for teaching.

With respect to future lines of research relating to this model, we feel that the most important thing is to hone the defining features of the subdomains, working with concrete examples which help to clarify each category, and to develop suitable methodological questions. An additional area of great interest is the question of what it is to know something, and we would like to undertake a comparison of different approaches to understanding what knowledge is (Meel, 2003), with the hope of arriving at a definition of knowledge in the specific case of a mathematics teacher that would allow us to study the relation between knowledge and conceptions.


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