DELINEATING ISSUES RELATED TO HORIZON CONTENT KNOWLEDGE FOR MATHEMATICS TEACHING

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The conceptualisation of mathematical knowledge for teaching (MKT) has recently received much attention. Scholars have provided examples, studied effects, and debated importance. However, from among the MKT domains, horizon content knowledge (HCK) has received less attention. In particular, the nature of the knowledge as it is related to teaching is unclear. We argue for efforts to clarify definitions and to test and refine those definitions with the use of realistic and vetted examples of professional work. To advance this agenda, we provide a working definition of HCK and use it to discuss a vignette involving irrational numbers.

INTRODUCTION

Teachers' content knowledge is of current interest, both the nature of such knowledge and ways to improve it. Among proposed conceptualisations, one that emerged from trying to understand and describe the nature and form of teaching and its mathematical demands is mathematical knowledge for teaching (MKT) (e.g., Ball, Thames, & Phelps, 2008). Grounded in Shulman's (1987) notions of subject matter knowledge (SMK) and pedagogical content knowledge (PCK), these researchers define MKT to be mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students and propose a refinement of SMK and PCK into sub-domains. Of these, horizon content knowledge (HCK) is less-developed.

The problem of defining HCK stems from an overabundance of metaphors and from inadequate clarity and consensus, especially regarding HCK's relation to teaching. Thus, deeper discussion of what HCK comprises (what it is and what it is not) is needed. Ball and Bass (2009) situate their conception of HCK within their practice-based theory of MKT. They describe HCK as "a kind of mathematical 'peripheral vision' needed in teaching, a view of the larger mathematical landscape that teaching requires" (p. 1). They provide a compelling foray into the ideas, but their provocative proposal leaves much to further development.

Starting with Ball and Bass' ideas, Zazkis and Mamolo (2011) use Husserl's work to propose a conception of "knowledge at the mathematical horizon." They use Husserl's notion to analyse ways in which topics from undergraduate mathematics provide inner and outer horizons of school mathematics. Their paper prompted two commentary papers. Foster (2011) discusses what he calls “peripheral mathematical knowledge” to refer to mathematics that matters for teaching but is out of the view of the learner. Figueiras, Ribeiro, Carrillo, Fernández and Deulofeu (2011) point out that the language for HCK needs to be consistent with basic assumptions of the nature and role of teacher content knowledge. They argue for locating the meaning of HCK in the work of teaching instead of conceptualising HCK as advanced knowledge that is then applied to teaching. They write, “Our critique of Zazkis and Mamolo’s paper is much more in terms of their assumptions about the nature of the mathematical knowledge that elementary and secondary teachers need, rather than in terms of their conceptualization of knowledge at the mathematical horizon” (p. 26). The point they seem to be making is about whether the knowledge from an advanced course would have the bearing on practice that Zazkis and Mamolo claim.

A different view of knowledge in relation to teaching is evident in the work of Vale, McAndrew, and Krishnan (2011). They use the phrase “connecting with the horizon” to characterize the knowledge implicated by teachers comments that learning more advanced mathematics in a professional learning program helped them see more connections and structure both among representations related to a topic and among different topics. They report that, as teachers saw connections between topics they teach and more advanced topics, it helped them see connections and more general structure inside the mathematics they teach.

In these studies, several images, phrases, and issues recur, often in ways that reveal unresolved issues. For example, some scholars identify with the language from Felix Klein’s book title, elementary mathematics from an advanced standpoint, or higher perspective, while others suggest inverting it to be advanced mathematics from an elementary perspective. As another example, references are sometimes to students’ horizons and other times to teachers’ horizons. In describing horizon knowledge issues also arise regarding distinctions between HCK and knowledge of the curriculum and its trajectory. Some scholars are concerned with the importance of requiring undergraduate mathematics courses, while others are concerned with the treatment of that content and the way in which it is framed and named.

We suggest that these scholars are engaged in a difficult process of developing a clear definition of HCK around which consensus could be built and that examples are central to this process. Explicit definitions, good examples, and disciplined analyses are crucial to making progress. In this paper, we offer our current “working definition” of HCK and then use it to examine a candidate example of HCK. We then use this examination to reconsider some of the issues central to HCK. Although situated in empirical work, our primary goal is conceptual — to surface, delineate, and clarify key issues for advancing work on HCK.

THEORETICAL BACKGROUND

Researchers have found that teachers' mathematical knowledge and experience, broadly construed, are not consistently associated with greater student learning. Instead, the mathematical knowledge associated with achievement gains is specifically related to the work of teaching and to the mathematical tasks that constitute that work. It is this evidence that led Ball and her colleagues to develop a conceptualization of mathematical knowledge for teaching, where the "for teaching" expression conveys a practice-based characterization of teacher content knowledge.

In addition to common knowledge of the subject (SMK), Shulman (1987) defines PCK as knowledge that is an amalgam of knowing the subject with knowing how students engage with the subject and knowing effective ways of representing the subject and rendering it for learning. Ball et al. (2008) subdivide both SMK and PCK. PCK contains: i) knowledge of content and students (KCS); ii) knowledge of content and teaching (KCT); and iii) knowledge about content and curriculum (KCC). SMK contains: i) common content knowledge (CCK), mathematical knowledge that is involved in teaching but not unique to the teaching profession; ii) specialized content knowledge (SCK), mathematical knowledge that is unique to teaching and not used in professions outside teaching (namely, knowledge that allows the teacher to engage in tasks specialized to teaching, such as analyzing patterns of errors or readily solving problems in multiple ways); and iii) horizon content knowledge (HCK), knowledge about mathematics outside the curriculum. An important contribution of the work of Ball and her colleagues is that all of these domains are defined in relation to the work of teaching: MKT is knowledge that serves as a resource for addressing the mathematical demands of teaching. It is this knowledge "for teaching," with clear links to the demands of specific tasks of teaching, has been shown to have positive effects on student achievement (Baumert et al. 2010; Hill, Rowan, & Ball, 2005; Kersting, Givvin, Sotelo, & Stigler, 2011; Rockoff, Jacob, Kane, & Staiger, 2008). In this discussion, it is also worth noting that, because MKT is intimately linked to teaching, it is different for different school levels and topics: MKT for kindergarten differs from MKT for upper elementary differs from MKT for secondary, and MKT for geometry differs for MKT for number and operation differs from MKT for algebra.

HCK is one of the sub-domains of such a practice-based mathematical knowledge for teaching. It involves a sense of how mathematics at play in instruction is related to a larger mathematical landscape (Ball & Bass, 2009). HCK is thus perceived as implicated by the proximal demands of teaching but not directly related to the curriculum (mathematical content) that has to be taught at a particular point in instruction. Importantly, even though it is about mathematical knowledge removed from the content being taught and learned at a particular level, HCK needs to be demonstratively related to the teaching that takes place in school.

In collaboration with Ball and Bass' research group at the University of Michigan, we developed the following working definition.

> Horizon Content Knowledge (HCK) is an orientation to and familiarity with the discipline (or disciplines) that contribute to the teaching of the school subject at hand, providing teachers with a sense for how the content being taught is situated in and connected to the broader disciplinary territory. HCK includes explicit knowledge of the ways of and tools for knowing in the discipline, the kinds of knowledge and their warrants, and where ideas come from and how "truth" or validity is established. HCK also includes awareness of core disciplinary orientations and values, and of major structures of the discipline. HCK enables teachers to "hear" students, to make judgments about the importance of particular ideas or questions, and to treat the discipline with integrity, all resources for balancing the fundamental task of connecting learners to a vast and highly developed field.

HCK is neither common nor specialized, and it is not about a curriculum progression, but more about having a sense of the larger mathematical environment of the discipline being taught. In that sense, when discussing HCK, it is not sufficient to simply consider knowledge about advanced mathematics or knowledge about different topics that may arise in students' future studies. HCK also includes, but not to the exclusion of other things, knowledge that would allow teachers to make additional sense of what students are saying and to act with an awareness of connections to topics that students' may or may not meet in the future.

HCK is distinct from specialized content knowledge (SCK) because SCK is immediately about the content being taught and HCK is not. SCK is about unpacked, elaborated, explicit versions of the content being taught, in ways that are useful to teachers as they teach. Beyond what students directly need to learn, it includes knowledge about representations, explanations, language, and features of these that increase teachers' capacity to teach them. The distinctive character of SCK is evident in Foster's (2011) discussion of peripheral mathematical knowledge. As he writes when discussing knowledge that he found useful as a teacher (such as knowing that both $x^2 + 17x + 30$ and $2x^2 + 17x + 30$ factorise with integer coefficients and how to generate other such pairs), "that whereas the process of *coming to know* these things may be of great value for learners, *knowing them* may not be" (p. 25).

The notion of SCK as distinctively mathematical knowledge directly related to the content being taught but that is specialized to the work of teaching has a certain parallel to the notion of an inner horizon as described by Zazkis and Mamolo (2011, p. 9) as not at the focus of attention, but also intended. If the last part were amended to "also present" or "also relevant to teaching," the parallel would be quite strong. It is this issue of relevance to teaching that seems to distinguish Zazkis and Mamolo's examples from those of Foster's. When we read Zazkis and Mamolo's example of knowing that the number of triangles in a pentagon with all diagonals drawn is a multiple of five, we can follow the logic that there might be a situation in which this might be relevant as a teacher, but it does not pass a kind of reality test for professional knowledge. At a practical level, in an extended discussion in the professional community, we are not convinced that this would be seen as having much utility, whereas many of Foster's examples seem likely to stand up to such a test. At the heart of this is the basic definitional character of MKT — that it be professional knowledge *for teaching*.

The other distinction to make is between HCK and KCC. We argue that it is important to distinguish these and that doing so reveals a central issue, that HCK is distinctively relevant to the conversation about "advanced" mathematics courses and the role of mathematicians in the education of teachers, whereas KCC is much more about an understanding of school mathematics and particular approaches to organizing the school curriculum. Unfortunately, Ball et al.'s description of HCK as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" confounds the issue. Their statement was meant to be about one example of HCK and the meaning of "related" was not meant to be about the curricular development of the content, but about the other kinds of relatedness that might exist among topics (personal communication). In this unfortunate wording, we see potential problems with the term "horizon." The idea of a curricular horizon as being about a curricular trajectory is distinct from what we mean by HCK. The statement also raises the issue of how remote something needs to be in order to be about the horizon. Our current proposal is that HCK is not the content being taught and not about curricular development of that content. Next we discuss the empirical context of our work and then present a vignette from secondary school teaching and use it to discuss the constitution of HCK.

METHODS

To establish a practice-based conception of mathematical knowledge at the horizon, we have grounded our analyses in teaching and practice-based reasoning about teaching, such as that occurring in the context of focus-group interviews, professional development programs, and collaborative investigation of teaching. The data consist of records of such practice from the United States, Norway, and Portugal. From this data, we selected and developed candidate, teaching vignettes — critical situations perceived as effective teaching and supporting professional deliberation, discussion, and discernment. We then analyze these vignettes in relation to our working definition of HCK and vet them with other members of the mathematics and mathematics education community.

Although we focus on practice, it is important to note that the object of study is not a particular teacher or classroom, but tasks entailed in teaching and an analysis of their mathematical demands (Ball & Bass, 2003). With a focus on idealized tasks of teaching and the demand they create for horizon knowledge, we developed vignettes from selected episodes gathered in different contexts. The vignettes are consistent with professional practice, which allows us to use them to examine and test notions of HCK. The vignettes provide a reference point for discussing, reflecting upon, and further analyzing both the work of teaching and our findings about practice-based

HCK. Given limited space, we discuss a single example, and then use it to illustrate and reflect on the proposed definition.

Vignette of HCK in practice

Mr. Lee's class has been discussing different types of numbers. While his students have a firm knowledge of whole and rational numbers, they have now been introduced to irrational numbers and are given examples of such numbers (like $\sqrt{2}$ and π, listed on the board). Based on what is on the board, Mr. Lee asks his student whether they can think of any other rational or irrational numbers they have learned about in the past. A student, Jay, suggests $2\sqrt{2}$. Writing it on the board, Mr. Lee asks, "And is $2\sqrt{2}$ rational or irrational?" To his surprise, Jay says it is rational. Jay continues and another student Ben responds.

Jay: If you have a rational number and an irrational number and you multiply them, the product will be a rational, and you will still have a fraction.

Ben: I don't think so, because when we multiplied a rational with an integer, we still got a rational — I think the same… the product of an irrational and a rational will be irrational.

Jay goes to the board to explain his thinking:

Jay: Look…, say you have a rational *a/b* and multiply it by the irrational *v*, you get *av/b*, which is rational, see?

Ben: No, that can't be…. If it's rational… that is only possible if *a/b* is zero, and that was not the case.

Jay: What? … how come?

Ben: Oh, now I understand why you're saying that it's rational… you were missing something… well…, if the product is rational, then *v* is rational too, and it can't be because we said at the beginning that it was irrational...

There are several issues here that teaching needs to handle. First of all, a teacher would need to decide if the argument provided by Ben is correct. Second, after understanding that Ben's solution is correct, a teacher would need to decide whether it is worth pursing, in particular when the argument used by Ben is unrelated to the learning goals of the lesson and is outside Mr. Lee's secondary curriculum. Asking students to explain their ideas to their peers may be risky when the mathematics is in an area that is less familiar to the teacher. These situations, corresponding to improvisations (Ribeiro, Monteiro, & Carrillo, 2009), lead to contingency moments (Rowland, Huckstep, & Thwaites, 2005) in which the teacher has to put in practice all of his or her intuitive knowledge. Even when teachers are aware of some of the possible implications in following, or not, a certain path, they face dilemmas that can profit from a familiarity with mathematics beyond the scope of what is being taught.

A teacher should notice that Jay's first argument is wrong. In the vignette, Mr. Lee does not intervene and lets the two students continue, possibly because Ben is providing an argument and is trying to generalize from previous experience — an integer multiplied by a rational number yielded a rational number — so the teacher lets the students continue without intervening. However, in responding to the (wrong) argument Jay presented at the board, Ben is using a kind of proof-by-contradiction argument, content that is outside of the curriculum Mr. Lee intends to teach. When Ben states that av/b being rational implies $a/b = 0$, he starts by assuming that a/b is not zero. Then $(a/b)(v) = (av)/b = p/q$ is a rational number, and because a/b is not zero, $v = (pb)/(aq)$, meaning v is a rational number too, but that contradicts the assumption. So, a/b cannot be different from zero if av/b is a rational number. Having knowledge about proof by contradiction and the characteristics of irrational numbers would help a teacher to understand Ben's argument and to handle the teaching of this matter with integrity. However, having learned about irrational numbers and proof by contradiction in advanced university courses in mathematics is not a warrant that a teacher can make sense of students' arguments in a classroom. What is needed instead is a treatment that accounts for how these topics may arise in teaching.

As stipulated in the definition, HCK includes an understanding of disciplinary ways of knowing and of establishing validity. Being familiar with proof by contradiction and other important ways of building mathematical arguments would help a teacher hear student reasoning, in the many different and emerging forms it takes, whether at the primary or secondary level. Such knowledge would allow a teacher to appreciate and understand how proof by contradiction can be related and imbedded in students' comments and reasoning in many topics. Proof (and in particular proof by contradiction) is a topic more extensively taught and used in university mathematics courses, yet it is rarely related to what is done in teaching (and not all advanced topics have such a relation — direct or indirect).

The decisions faced by Mr. Lee require subject matter knowledge not part of the curriculum he is teaching. Hence, it is not obviously part of SCK or CCK, which involve representations, explanations, and unpacked knowledge of the content being taught. Knowledge of proof by contradiction and being able to use this knowledge in teaching, can help a teacher hear students, see beyond the topics being taught, and make judgments about what to do in the unfolding dynamics of instruction.

SUMMARY AND REFLECTING COMMENTS

We have used this example to examine a view of advanced mathematical knowledge related to teaching and how it is different from advanced mathematical knowledge typically taught to prospective teachers. Knowledge of advanced content is in itself not a warrant that a teacher can make sense of student thinking in instruction.

Returning to our working definition of HCK, teachers need to have an "orientation to" and "familiarity with the discipline." It would be helpful for teachers who are working with specific content to know how the discipline handles this content at different stages in its development. A topic can be related to other content, outside the immediate curriculum, with different aims and not directly related to the content being taught. Proof by contradiction is often taught in the context of number theory, yet familiarity with it can have a bearing on the teaching of a wide range of topics. Thus, teachers need knowledge that supports connecting the notion of a proof technique, and the mathematical steps of such a proof, with what students may say. This lead us back to our working definition, which says that HCK "contributes to the teaching at hand, providing teachers with a sense for how the content being taught is situated in and connected to the broader disciplinary territory."

Even though the reasoning of the students may be considered to be outside the curriculum, such as in the vignette above, a teacher may value the way students make use of specific proof techniques in their arguments, without necessarily trying to make the logic behind the proof an important part of the discussion. Instead, a teacher may recognize a proof technique, its validity, and the intuitive way it convinces students. Such knowledge is an important resource for teaching.

Our view is that teachers need a treatment of "advanced" mathematics tailored to the orienting and navigating demands of the teaching in which they engage. Experiences with proof in general, and proof by contradiction in particular, in the context of teaching, provide a teacher with resources for hearing the mathematical ideas behind Ben's argument — ideas related to major structures and developments in the discipline, another part of our working definition of HCK. A teacher needs to make decisions about how to handle discussions that occur in the classroom, and to do so in a way that has integrity as students learn additional mathematics. Knowing mathematics at the horizon gives a teacher awareness of potentialities of situations and suggests possibilities for dealing with the mathematical content being taught at a given level. In order to do so, a teacher does not need to know everything about proof by contradiction, but needs to have a sense of what it is and its potential. That would allow students (at least in theory) to further understand and make sense of other topics, both directly and indirectly related.

Developing tasks related to advanced content, yet situated in artifacts from teaching — such as student work, a task from a textbook, or a dialog among students — might motivate prospective teachers to study mathematics, provide a focus for learning that mathematics, and develop a sense of when and how to use such knowledge in teaching. Similarly, developing instruments to measure HCK might help advance our understanding of HCK by forcing greater clarity about what it is we are trying to measure. Instruments could be similar to many of the recent multiple-choice instruments developed to measure other domains of mathematical knowledge for teaching (e.g., Hill, Schilling, & Ball, 2004; Tatto et. al, 2008), or they could be developed using interviews or observational techniques. Indeed, measuring and validating the models underlying such concepts would be important steps in developing an understanding that could inform policy and practice.

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