THE ROLE OF DIDACTICAL KNOWLEDGE IN SEIZING TEACHABLE MOMENTS[1]

Rosa A. Tomás Ferreira*, Maria Helena Martinho**, & Luís Menezes***

*University of Porto & CMUP, **University of Minho & CIEd, ***CI&DETS & College of Education of Viseu – Portugal

Drawing on a short analysis of a classroom episode, we reflect on the teacher's actions and their relationship to her didactical knowledge, namely in its dimensions of knowledge of mathematics and knowledge of instructional processes. Focusing on these dimensions, we discuss the answers of some prospective and practicing teachers to a written assignment based on that episode. Anchored in the notion of didactical knowledge, we raise some issues regarding teacher education programs and their adequacy to comply with current demands of mathematics teaching.

TEACHERS' DIDACTICAL KNOWLEDGE AND CLASSROOM EPISODES

Portugal's recent mathematics curriculum for basic education (grades 1 to 9, pupils aged 6 to 14) (Ministério da Educação (ME), 2007) stresses three transversal skills – problem solving, reasoning, and communication – which are seen of crucial importance towards achieving the curriculum overarching learning goals. However, we believe that the recommended changes in the dynamics of the mathematics classroom are the crucial features which put the biggest challenges to teachers. Indeed, teachers and their students are called to play very active roles within mathematically rich environments.

In our work with future and practicing mathematics teachers, we pay special attention to issues of classroom communication, stressing the teacher's role in the process (Bishop & Goffree, 1986; Menezes, 2004; Tomás Ferreira, 2005; Martinho & Ponte, 2009; Ruthven, Hofmann & Mercer, 2011). The analysis and discussion of short classroom episodes – written vignettes of lesson snapshots – is a way that has been found to be useful in helping teachers recognize situations which illustrate challenges that they find when engaging students in meaningful mathematical discourse (Ruthven et al., 2011; Tomás Ferreira, Menezes & Martinho, 2012). Having a sound didactical knowledge seems to be of utmost importance to attain such goal.

Though acknowledging other interpretations for the idea of didactical knowledge (Ponte, 2012), we follow Ponte's (1999) perspective in which it is directly related to aspects of practice and is "essentially oriented towards the action" (p. 61). The notion of didactical knowledge encompasses four inter-related dimensions:

(1) *knowledge of the content* that is to be taught, including connections amongst mathematical concepts and connections with other areas and their reasoning, argumentation, and validation forms;

(2) *knowledge of the curriculum*, its goals and objectives, and its horizontal and vertical articulation/alignment;

(3) *knowledge of the students*, their learning processes, interests, and most frequent needs and difficulties, as well as knowledge of social and cultural factors that may influence students' performance at school; and

(4) *knowledge of the instructional process*, namely the planning and teaching of lessons, and the assessment of teachers' own practices. (Ponte, 1999, p. 61 [italics added])

This notion also involves knowledge of the contexts (e.g., school, community) and knowledge of self as a teacher (Ponte, 2012). Didactical knowledge is dynamic in nature since "the experiences and situations of practice the teacher encounters in the classroom contribute to its development and constant reformulation" (Tomás Ferreira et al., 2012, p. 283; see also Ponte & Santos, 1998).

The study reported in this paper emerged from our practice as mathematics educators. We start by analyzing a classroom episode, discussing some aspects of the teacher's didactical knowledge that support her core actions. We then present the analyses of that episode made by prospective and practicing teachers, discussing aspects of their didactical knowledge regarding the domains of *knowledge of mathematics* and *knowledge of instructional processes*. Finally, we share some thoughts about teachers' didactical knowledge and raise issues about teacher education programs and their adequacy to comply with current demands of mathematics teaching.

A Classroom Episode

In the episode *Rita and Prime Numbers* (Figure 1; Boavida, 2001, adapted from Prince, 1998), the teacher starts by proposing a closed task – to list all prime numbers up to 50 – which has a low level of cognitive demand for her students (the students in the episode correspond to Portuguese 7th graders, aged around 12). Yet, by building on a student's (Rita) comment, the teacher raised the task's cognitive demand, engaging the students in complex thinking processes, such as conjecturing, refuting, arguing, and proving. In addition, they have the opportunity to discuss aspects of basic logics (such as implications, reciprocals, examples, and counterexamples).

Rita's teacher asked her class to **find all prime numbers up to 50**. After some time, Rita noticed that the prime numbers larger than 5 she had identified so far ended in 1, 3, 7, or 9. She called her teacher to show her this finding. The teacher asked Rita to work with her partner in order to find the best way to share her finding to the class during the collective discussion of the work. Rita listed on the board all prime numbers smaller than 50 and she read what she had written in her notebook:

Rita: The prime numbers except 2 and 5 end in 1, 3, 7, or 9.

The teacher then asked the class to analyse if the same thing happened with other prime numbers. The students started checking several cases of prime numbers, some of which much larger than 100, and they did not find anyone that would not end in 1, 3, 7, or 9. Shortly, they were strongly convinced that what Rita had found was true for all prime numbers, regardless of having been checked, because all prime numbers that they would check always ended in one of those digits. At this time, the teacher wrote on the board:

Rita's conjecture: ***All prime numbers, except 2 and 5, end in 1, 3, 7 or 9.***

She made sure the students remembered the meaning of conjecture and she challenged them to find a process that would allow them to be sure if the conjecture were, indeed, valid for all prime numbers and why that was so. The students tried to respond to the challenge and, in this process, they reinforced their conviction that the conjecture was true; yet, their work did not progress. Then working with the whole class, the teacher wrote on the board the numbers from 0 to 9, circling 1, 3, 7, and 9. Almost immediately the students offered several suggestions:

Maria: Teacher, cross out numbers 0 and 5. A prime number larger than 5 cannot end in 0 or 5.

Teacher: Why?

Maria: If it ends in 0 or 5, it is a multiple of 5 and, therefore, it is not prime.
Daniel: You also have to cross out 2, 4, 6, and 8. If it is larger than 2 and is a prime number, it cannot be even!
Bernardo: Of course not. 2 divides...
Teacher: So what?
Bernardo: A prime number can only have two factors.
Rosa: Yeah. If it ends in 2, 4, 6, or 8, that is because it is even and even numbers are multiples of 2.
Rita: We only have 1, 3, 7, and 9 left. Therefore, all prime numbers except 2 and 5 end in the way I found. We're now sure of it.
Inês: **But the opposite is not true.** For example, 21 ends in 1 and it is not prime.
Teacher: Why isn't 21 prime?
Several students: Because 3 divides 21; 3 times 7 is 21.
Bernardo: It has divisors that are different from 1 and 21.
Teacher: So, check if this is true, or not, this that I am going to write on the board: all numbers ending in 1, 3, 7, or 9 are prime.

One could hear several voices saying "it's not true". They mentioned 21, 27, 33... (…)

Figure 1. Episode *Rita and Prime Numbers*

Which instructional actions are central in the unfolding of this episode? The teacher refrained from validating Rita's idea; instead, she gave Rita and her colleague the opportunity to present their finding to the class. By building on Rita's input, the teacher extended the original task, asking the class to check whether Rita's idea would work for other prime numbers. The class naturally accepted its truthfulness as students were unable to find a way to contradict Rita's finding. The writing of "Rita's conjecture: All prime numbers, except 2 and 5, end in 1, 3, 7, or 9" on the board seems to have been deliberate – the teacher knew that the proof of a conjecture and the role of examples in that process were at stake, and that the term "conjecture" could be unfamiliar to some students. In addition, she turned Rita's conjecture more explicit to the whole class by clarifying its scope.

The collective discussion that was initiated engaged students in reflecting on the meaning of conjecture and of proving or refuting a conjecture. This was not an easy task for the students who could only see their ideas reinforced by finding more and more examples which, nevertheless, proved nothing. After letting the students struggle with this, the teacher wrote on the board all ten digits and circled those corresponding to the units of a prime number – her intention seems to have been to list all possibilities for ending a natural number while highlighting those related to Rita's conjecture. Drawing on their knowledge of divisibility criteria, the students eliminated the non-circled digits and the teacher's questions ("Why?"; "So what?") ensured that they justified all their options. By building again on a student's comment (that the opposite of Rita's conjecture was not true), the teacher involved the students in working with counterexamples, which started to emerge after she wrote the new conjecture on the board. In sum, the teacher's actions raised the cognitive level of the initial task and helped engaging the students in significant mathematical activity.

We can identify some aspects of the teacher's didactical knowledge in the episode *Rita and Prime Numbers*. For example, the teacher listened to her students in a responsive manner (Empson & Jacobs, 2008), valuing all contributions as worthy discussing collectively, regardless of their correctness or rigorousness in language. By giving the students the responsibility for proving or refuting the two conjectures presented in the episode, the teacher orchestrated a collective discussion in a productive manner (Stein, Engle, Smith & Hughes, 2008), pushing for a shared understanding of conjecture and for explanation and justification of all assertions.

We believe the teacher's actions were anchored in her *mathematical knowledge*, which allowed her to recognize a teachable moment triggered by Rita's finding, and in her *instructional knowledge*, which allowed her to seize the situation and build instruction upon Rita's idea, encouraging her students to do mathematics (Tomás Ferreira et al., 2012). The teacher transformed a task of procedures without connections (Stein & Smith, 1998) into a task with much higher cognitive demand, involving processes of proof. The new task and the fruitful discussion around it pushed the students to engage in significant mathematical activity.

DATA GATHERING

Our practice as teacher educators reflects our belief that is it important to have (prospective) teachers discussing aspects of the teacher's role regarding the management of mathematical communication in the classroom. For that purpose, we frequently resort to the analysis of classroom episodes such as the one presented before. The data we present and discuss next is based on the analysis of the episode *Rita and Prime Numbers* guided by the following questions: (1) *How do you think the teacher should lead the classroom discourse after the last interventions of the students?* and (2) *Do you believe Rita's conjecture is proved? If so, why? If not, why?* At the end of 2011/11, a group of 12 prospective teachers, enrolled in a 2-year master's teacher certification program, was asked to complete a short written, individual, in-class assignment which included the analysis of the episode *Rita and Prime Numbers* and accounted for 10% of their final grade. There was great variation in the answers obtained not only regarding the mathematics underneath the episode but also in terms of the didactical choices that were thought to be adequate to give continuation to the episode.

Feeling the need to see practicing teachers' reactions, we asked a group of eight teachers, enrolled in a professional development course, to analyse the same episode, using the same guiding questions; yet, the assignment did not count explicitly for assessment purposes. The two cohorts of participants worked in different universities located in large urban areas in northern Portugal; in both contexts, participants had been involved in reflecting and discussing several issues of communication, especially the teacher's role in managing meaningful classroom discourse (National Council of Teachers of Mathematics (NCTM), 1991), and the challenges faced when orchestrating productive mathematical discussions (Stein et al., 2008). The participants' written productions were analysed in the light of Ponte's (1999) notion of didactical knowledge, focusing on the dimensions of mathematical and instructional knowledge.

RESULTS

In this section, we present and analyse some of the data collected from both groups of participants, resorting to our translation of the participants' work because it is originally written in Portuguese. We chose the work of three prospective and two practicing teachers as it illustrates the respective cohort productions. We structure our discussion by each of the two questions that guided the analysis of the episode.

How Should The Episode Continue?

Júlio held a bachelor degree in mathematics from the same institution he was seeking teacher certification. In his response to the first question there was evidence that he acknowledged the existence of two implications in the episode, one being the reciprocal of the other. Focusing on sense making and knowledge building, he emphasized the need of recognizing and distinguishing reciprocal implications, and understanding the role of examples and counterexamples in proofs and refutations:

> Based on the students' answer[s], the teacher should tell them that they had shown the assertion was false, through a whole-class discussion, making them understand that it is enough to give an example that does not verify the assertion for this to be invalid. Then, she should ask the students to relate Rita's conjecture to the latter one, questioning them about their difference[s] and truthfulness, in order to conclude the task.

Júlio's sensitiveness towards the important issue of developing mathematical reasoning, particularly formulating, testing, and proving (or disproving) conjectures, in the teaching and learning process seemed to be clear.

Carlos was a colleague of Júlio's, with a similar academic background. Unlike Júlio, Carlos did not evidence much understanding about the episode, due to an incorrect interpretation of the episode or to weaknesses in his didactical knowledge. His suggestion to continue the episode began with some considerations about Rita's finding, which evidence fragilities in his mathematical knowledge:

> The way Rita phrased the conjecture seems to indicate that all prime numbers are all odd numbers except those that end in 5. During the lesson, it became clear that this is not true since 21, 27, 33 are odd numbers ending in 1, 7, and 3, and they are not prime.

This prospective teacher did not seem to realize that the discussion at the end of the episode was about the reciprocal of Rita's conjecture and that the examples provided by the students (21, 27, and 33) were, indeed, counterexamples for the reverse of Rita's conjecture, not counterexamples for the conjecture itself. Besides a poor understanding of the mathematical situation underlying the episode, Carlos's suggestions to continue the episode missed some important points emphasized in current curricular orientations:

> After the students said that it was not true, that all prime numbers end in 1, 3, 7, or 9, the teacher should ask them for explanations. Some mention examples that do not verify the conjecture; yet, the teacher should ask for more examples and have them discussing the

> reason why they are not prime [numbers]. Afterwards, [the teacher] could build on the fact that 9 is not prime since the conjecture said that all numbers ending in 9 were prime.

Carlos did not assign an appropriate value to having students understanding the meaning of conjectures, reciprocals, examples, counterexamples, proofs, refutations, etc. (at the level of 7th graders), nor to having a moment in the lesson to summarize the ideas that emerged during class discussion. In addition, it was not clear why, according to Carlos, the teacher should deal with the number 9 in a *special way*. Data suggested that Carlos had a poor mathematical understanding of the episode.

Joana had earned a bachelor degree in applied mathematics and computing several years before enrolling in a teacher certification program. She worked in the field of applied mathematics and had a very short teaching experience. Her knowledge of mathematics exhibited several weaknesses, which seemed to account for inadequate instructional decisions. She misunderstood Rita's conjecture and its reciprocal; hence, not surprisingly, her suggestions to continue the episode seemed to be senseless:

> The teacher should have let the students reach the conclusion that 'all numbers ending in 1, 3, 7, 9' are not prime and she should not have written on the board and telling the conclusion. Maybe saying the students should conclude or even writing only the sentence 'all numbers ending in 1, 3, 7, or 9 are prime; do you agree?' because, by saying 'So, see if it is true' she is implicitly telling the students that something is wrong.

Joana did not interpret the teacher's intentions when writing on the board the two implications as a means to help students understanding what was at stake and to help them differentiating the two situations; instead, she saw the teacher's actions as intending to offer the students with clues for what would be correct or incorrect.

Cláudia was a certified teacher with more than ten years of experience teaching 7th to 9th graders (ages 12 to 14). In her response to the first question, she stressed critical features for continuing the episode: she believed that "at the end, the teacher would probably let the students prove that not all numbers ending in those [units] digits are prime [numbers]", giving the students accountability and ownership for drawing an important conclusion based on their own (counter)examples; in addition, she reinforced the need for a moment of synthesis – "in the end, it is important that they make a synthesis" – making explicit the main ideas that had been discussed.

Lina's teaching experience was similar to Cláudia's but she worked with 5th and 6th graders instead. Lina might have not understood what was being discussed at the end of the episode. She believed that "after the last interventions […] the teacher should ask the students to reformulate the conjecture". Her response to the second question (which we discuss later) sheds more light into her thinking.

Is The Conjecture Proved or Not? Why?

Júlio understood that Rita's conjecture was true and realized the process that was used to prove it collectively: "the conjecture is proved, since the students know that all the numbers end on some digit between 0 and 9, and using divisibility criteria, they managed to exclude the even digits and the 5, remaining 1, 3, 7, and 9". The data suggest that Júlio valued the whole-class discussion and students' (prior) knowledge as means to help them proving Rita's conjecture: "In this way, and using their own knowledge, the students proved Rita's conjecture, through discussion and exchange of ideas". Thus, Júlio seemed to have pulled adequate aspects of instructional knowledge to the analysis of the episode *Rita and Prime Numbers*.

Carlos believed that Rita's "conjecture may lead to two interpretations, the first being that all prime numbers are all [numbers] that end in 1, 3, 7, and 9, except 2 and 5, and, on the other hand, that all prime numbers except 2 and 5 end in 1, 3, 7, or 9". Although this latter interpretation was, in fact, Rita's conjecture, it was the first interpretation that Carlos believed to be at the core of the episode. He did not think that Rita's conjecture was proved during the lesson: "Rita's conjecture was not proved since prime numbers except 2 and 5 end in 1, 3, 7, [or] 9. What was proved was that the numbers ending in 1, 3, 7, and 9 aren't always prime". Resorting to the two possible interpretations of Rita's conjecture he identified, Carlos stressed that "in [this] lesson, the only thing that was proved was that the first interpretation is not valid". Data seemed to suggest that Carlos did not recognize a process of proof (of whatever conjecture he would consider) in the teacher's and students' joint work.

Joana did not understand Rita's conjecture *per se* and, in fact, it seemed that she had no understanding of what a conjecture is, nor what might be entailed in proving (or refuting) such an assertion:

> Rita's conjecture was proved and it was incorrect, since the students checked for a large array of numbers, even bigger than 100, thus establishing a degree of certainty in their answers and even finding numbers like, for example '21' which though ending in 1 is not prime since 3 divides 21.

Joana confused the two reverse implications involved in the episode; yet, she seemed to value the testing process as important to strengthen one's conviction.

Cláudia believed that Rita's conjecture was proved during the lesson; yet her response may indicate how easily teachers do what they say should not be done! In fact, she referred that "…although the student [Rita] had said 'We're now sure of it', she [the teacher] should show that there were prime numbers ending in 1, 3, 7, and 9". Cláudia showed concern for illustrating the core idea that was being discussed (Rita's conjecture) with concrete examples, which she seemed to believe that would help in reassuring the validity of the conjecture or in better understanding the conjecture. Though understandable, such a concern and subsequent actions may actually induce students into an erroneous conception of proof, namely mistaking proof for exemplification using particular cases.

Lina did not seem to be sure about whether Rita's conjecture had or had not been proved; in fact, after stating that the conjecture had been proved, she changed her opinion supported in a mathematically incorrect argument. Like Carlos, Lina brought up the number 9 into the scene, suggesting that she also had an unclear understanding of the conjecture that was collectively proved during the lesson:

> It seems that Rita's conjecture is proved because valid arguments were used allowing to conclude that the assertion is valid for all prime numbers except 2 and 5. However, the number 9 is not a prime number. There is at least one exception that was not considered; thus, the conjecture is not proved. Proof, in mathematics, entails demonstration.

On one hand, Lina believed that the conjecture had been proved but, on the other hand, the *proof* that was made during the lesson was not enough! Data suggested that Lina held a rigid and formal perspective of proof and made contradicting assertions since a logical chain of valid arguments no longer seemed to be at the core of a proof.

CONCLUDING REMARKS

The participants' difficulties in analyzing the episode *Rita and Prime Numbers* seemed to be anchored in a poor knowledge of the mathematics involved. The differences in academic background of the prospective teachers may have accounted for the differences in the mathematical knowledge they evidenced. However, caution must be exercised. Carlos, whose background and grade point average was similar to Júlio's, also showed gaps in his mathematical knowledge. Joana, unlike her colleagues, did have some teaching experience; yet, her knowledge of instructional processes in the classroom emerged as much weaker than that of Júlio or Carlos.

An inadequate understanding of proof (in Cláudia's case) or a very rigid and formal conception of proof (in Lina's case) may also have been at the origin of the difficulties found whilst analysing the episode. It was possible to find, in both groups of participants, illustrations of misunderstandings regarding the role of examples and counterexamples in proving or refuting assertions; yet, only in the participant practicing teachers did we find clear evidence of closed conceptions about proof, which may have hindered them from recognizing a process of proof in the episode.

The gathered data suggest that a poor knowledge of mathematics on the (future) teachers' part seems to be associated with a weakened instructional knowledge. This supports the claim that adequate instructional decisions can hardly be made when teachers do not have a deep understanding of the underlying mathematics of teaching situations (Kahan, Cooper & Bethea, 2003). In particular, the orchestration of productive mathematical discussions and the systematization of (new) knowledge, two complex communicative actions (Menezes, Canavarro & Oliveira, 2012; Stein et al., 2008) and essential aspects of the teacher's role within the current (Portuguese) curricular orientations, cannot be appropriately approached if the teacher's knowledge of the mathematics underneath the teaching situation is not sound (Martinho & Ponte, 2009; Ponte, 2012; Tomás Ferreira et al., 2012).

Teachers need solid mathematical and instructional knowledge, in Ponte's (1999) sense, to be able to build on teachable moments such as the one triggered by Rita's contribution and, more generally, respond adequately to the many demands of classroom teaching. Yet, the current typical organization of teacher education may be failing to develop (future) teachers' didactical knowledge, including mathematical knowledge, despite a significant emphasis on content courses, especially in many teacher certification programs. In addition, (prospective) teachers may not be developing adequate conceptions of proof, aligned with current recommendations for school mathematics, which go much beyond processes such as two-column proofs.

Despite a heavy content load in the participants' academic background and despite all the emphasis put in the teacher's role in managing meaningful classroom mathematical communication in the two contexts in which the participants worked, we were surprised to see how much difficulty they had in making sense of the episode and in reflecting upon it in the light of current curriculum orientations. We believe that the discussion of concrete situations, based upon classroom episodes as the one presented in this paper, may contribute to teachers' increasing consciousness about their conceptions and practices, helping them in recognizing teachable moments and in building on them, seizing up the opportunities that emerge during classroom interaction and taking the most of them. But this is obviously insufficient.

We had no opportunity to interview the participants in this study in order to have a better grasp of their mathematical and instructional knowledge. Our data provides only a limited and short glimpse of what might be happening. Further research is needed to address more deeply how (future) teachers manifest their didactical knowledge and how teacher education and professional development programs may help them in developing that kind of professional knowledge.

NOTES

1. This work is supported by national funds through FCT – Fundação para a Ciência e Tecnologia, under the projects *Professional Practices of Teachers of Mathematics* (Grant PTDC/CPE-CED/098931/2008) and PEst-C/MAT/UI0144/2011, and by FEDER funds through COMPETE.

REFERENCES

Bishop, A. J., & Goffree, F. (1986). Classroom organization and dynamics. In B. Christiansen, A. G. Howson, & M. Otte (Eds.), *Perspectives on mathematics education* (pp. 309-365). Dordrecht: Reidel.

Boavida, A. (2001). Um olhar sobre o ensino da demonstração em Matemática. *Educação e Matemática, 63*, 11-15.

Empson, S. B., & Jacobs, V. R. (2008). Learning to listen to children's mathematical thinking. In D. Tirosh & T. Wood (Eds.), *The international handbook of mathematics teacher education* (Vol. 2, pp. 251-281). Rotherdam, the Netherlands: Sense Publishers.

Kahan, J. A., Cooper, D. A., & Bethea, K. A. (2003). The role of mathematics teachers' content knowledge in their teaching: A framework for research applied to a study of student teachers. *Journal of Mathematics Teacher Education, 6*(3), 223-252.

Martinho, M. H., & Ponte, J. P. (2009). Communication in the classroom: Practice and reflection of a mathematics teacher. *Quaderni di Ricerca in Didattica (Matematica)*, Supplemento n. 2 al n. 19.

Menezes, L. (2004). *Investigar para ensinar Matemática: Contributos de um projecto de investigação colaborativa para o desenvolvimento profissional de professores*. Doctoral dissertation. University of Lisbon, Portugal.

Menezes, L., Canavarro, A. P., & Oliveira, H. (2012). Teacher practice in an inquiry-based mathematics classroom. *HMS i JME - International Journal for Mathematics in Education*, Vol. 4, 357-362.

ME (2007). *Programa de matemática do ensino básico*. Lisboa, Portugal: DGIDC.

NCTM (1991). *Professional standards for school mathematics*. Reston, VA. NCTM.

Ponte, J. P. (1999). Didácticas específicas e construção do conhecimento profissional. In J. Tavares, A. Pereira, A. Pedro & H. Sá (Eds.), *Investigar e formar em educação – Proceedings of XI EIEM* (pp. 59-72). Porto, Portugal: SPCE.

Ponte, J. P. (2012). Estudando o conhecimento e o desenvolvimento profissional do professor de matemática. In N. Planas (Ed.), *Teoría, critica y prática de la educación matemática*. Barcelona: Graó.

Ponte, J. P., & Santos, L. (1998). Práticas lectivas num contexto de reforma curricular. *Quadrante, 7*(1), 3-33.

Ruthven, K., Hofmann, R., & Mercer, N. (2011). A dialogic approach to plenary problem synthesis. *Proceedings of the 35th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 81-88). Ankara, Turkey: PME.

Stein, M. K., & Smith. M. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle School, 3*(4), 268-275.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*(4), 313-340.

Tomás Ferreira, R. A. (2005). *Portuguese student teachers' evolving teaching modes: A modified teacher development experiment*. Doctoral dissertation. Illinois State University, USA.

Tomás Ferreira, R. A., Menezes, L., & Martinho, M. H. (2012). Using classroom episodes to foster prospective teachers' didactical knowledge: Issues for teacher education. In J. Dindyal, L. P. Cheng & S. F. Ng (Eds.), *Mathematics education: Expanding horizons* (Proceedings of the 35th annual conference of the Mathematics Education Research Group of Australasia, eBook, pp. 282-289). Singapore: MERGA, Inc.