A THEORETICAL CONSTRUCT TO ANALYZE THE THEACHER'S ROLE DURING INTRODUCTORY ACTIVITIES TO ALGEBRAIC MODELLING

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Abstract

In this work we will introduce a theoretical construct that we have elaborated as a tool for the analysis and the interpretation of the teachers' actions during class activities which are aimed at fostering an aware learning of the use of algebraic language as a thinking tool. Through the analysis of an excerpt of a class discussion concerning introductory activities to algebraic modelling, we will show how this construct could provide "transparent" indicators to highlight the effectiveness of the teachers' actions during class interaction.

1. INTRODUCTION

The idea of an early approach to algebra, with a strong focus on generational activities (Kieran 1996) to help students overcome the well-known difficulties they usually face in the study of the formal aspects of algebra, is widespread and consolidated (Carpenter&Al. 2003, Kaput&Al. 2007, Cai&Knut 2011).

Starting from the 90s, these ideas are developed by research together with a new vision of the teaching of arithmetic, characterised by a focus on relational aspects and meta-level activities, aimed at making students control the properties subtended to arithmetical equalities in order to create a connection between arithmetic and algebra (Bell 1996, Kieran 1996, Lincevski 1995). Althoug in the first decade of 2000 many research studies are devoted to the implementation of these activities at school (also at the primary level), only few of them consider the role played by the teacher together with the problem of teacher education (see for example Carpenter&Franke 2001, Blanton&Kaput 2001).

Our research studies, that can be conceived within this frame, are devoted to the planning of innovative didactical paths in arithmetic and algebra (grades 4-8) to be implemented through a socio-constructive approach in a strict cooperation with the teachers (Malara&Navarra 2003). The possibility to cooperate with different teachers (while in the first period we only collaborated with teacher-researcher, in these last ten years many other motivated and experienced teachers were involved) enabled us to highlight two main gaps: (1) a gap between theachers' declared conceptions and the hidden ones displayed by their behaviours in the classes; and (2) a gap between the theoretical assumptions they shared with researchers and their actual practice (Malara 2003). These results suggested us to focus not only on class experimentations but also on teacher education activities. In tune with Mason (1998) and Jaworski (2003) ideas, we designed and implemented specific specific tools and methods aimed at fostering teachers' development of different levels of awareness through the activation of joint critical-reflection practices. These tools and methods, have proved to be effective ways of fostering teachers' real professional development

(Cusi,Malara&Navarra 2011).
Our actual research objective is to design a specific methodological tool aimed at guiding teachers in the fundamental process of a-posteriori reflection on their own practice. Our idea is to refer to a theoretical construct, that has been defined as a result of one of our studies (Cusi&Malara 2009, Cusi 2012), as both a diagnostic tool in the analysis of class processes and a tool for teachers' self-reflection on their own teaching: the construct of "*model of aware and effective attitudes and behaviours*" (in the following M-$_{AE}$AB). Although it was aimed at identifying the specific features of a teacher who is able to make his/her students develop fundamental competences in the use of algebraic language as a tool for thinking (specifically, in the realm of proof), we believe that this construct could represent an effective "theoretical lens" for the analysis of the role played by the teacher, even during different algebraic activities, such as those aimed at the introduction of algebraic modelling.
In the following, we will introduce the M-$_{AE}$AB construct in the theoretical frame within which it has been developed and, through the analysis of a class excerpt, we will show how it could be refered to as a tool for analyzing the role of the teacher during introductory activities to algebraic modelling, highlighting its effectiveness in providing "transparent" indicators to describe the teachers' actions.

2. THE M-$_{AE}$AB CONSTRUCT FOR THE ANALYSIS OF THE TEACHER'S ROLE

The theoretical frame within which the M-$_{AE}$AB construct has been developed is constituted by two threesomes of components. The *first threesome* refers to the theoretical components we identified for the analysis of the development of thinking processes through algebraic language: (a) the model of didactic of algebra as a thinking tool proposed by Arzarello&Al. (2001), who, in particular, highlight the essential role played by the activation of conceptual frames and appropriate changes from a frame to another for a correct interpretation of the algebraic expressions which are progressively constructed; (b) the idea of anticipating thought developed by Boero (2001), who introduces it as a key-element in the "game" transformation-interpretation, which is typical of the processes of construction of reasoning through algebraic language; (c) the theoretical analysis proposed by Duval (2006), who identifies in the coordination between different representation registers a critical aspect in the development of learning in mathematics.
Thanks to previous studies (Cusi 2009) we were able to highlight that an effective use of algebraic language as a thinking tool requires the management of three main key-components: (a) the appropriate application of conceptual frames and coordination between different frames; (b) the application of appropriate anticipating thoughts; and (c) the coordination between algebraic and verbal registers (on both translational and interpretative levels).
The *second threesome* of components is related to our theoretical framework of approach to the study of the teaching-learning processes and of the role played by the teacher. The first component is Vygotskian: we, in particular, refer to Vygotsky's stress (1978) on the importance of a teaching aimed at expanding students' zone of

proximal development in order to stimulate, thanks to their interaction with the teacher or with more expert classmates, the activation of internal learning processes associated to a higher level of mental development. The second component draws its inspiration from the work carried out by Leont'ev (1978), who stresses the importance of making students increase their awareness about the meaning of the processes they activate during class activities in order to foster their learning. Our third component is the cognitive apprenticeship model introduced by Collins & Al. (1989), which draws its inspiration from an idea of learning as an "aware" apprenticeship and pursues the objective of "making thinking visible", through the activation of teaching methods which give students the opportunity of observing, discovering or even inventing the experts' strategies in the same context in which they are worked out. Specifically, we refer to two sets of typical methods of cognitive apprenticeship: (a) modeling, coaching and scaffolding, aimed at helping students acquire skills through processes of observation and guided practice; (b) articulation and reflection, related to metacognitive objectives and aimed at helping students achieve a conscious control of their own problem-solving strategies.

We believe that the 'games' of coordination between different linguistic registers and of interaction between the syntactical level, the interpretative level and the level of activation of anticipating thoughts, which can be automatically set up by an expert, should be "made visible" to novices in order to make them acquire and understand their meaning. Therefore our hypothesis is that, in order to help students progressively develop the competences and awareness necessary to carry out advanced tasks through an effective use of algebraic language, it is necessary that the teacher, during class interaction, adopts and makes visible specific attitudes and behaviours. In this way, his/her students could be guided to the acquisition of the same attitudes and behaviours, which corresponds to an effective management of the three key-components we have previously introduced.

This is the reason why we decided to refer to the expression "*teacher as a model of aware and effective attitudes and behaviours*" to highlight the approach of a teacher who consciously behave with the constant objective of "making thinking visible", in order to make his/her students focus not only on syntactical aspects but also on the effective strategies and on the meta-reflections on the actions which are performed.

In order to identify the peculiar characters of a teacher able to adopt this kind of approach in the class, during previous studies (Cusi&Malara 2009; Cusi 2012) we analyzed the audio-recordings of whole class activities and the subsequent students' small-groups activities, with the aim of highlighting: (1) the role played by the teacher during class activities as a "stimulus" to foster an approach to algebra as a tool for thinking, and, (2) the links between the types of approach proposed by the teacher and the types of approach chosen by students during small-groups activities, with particular reference to the meta-reflections they propose. This analysis enabled us: (1) on the one hand, to highlight how unsuitable teacher's choices can lead to a missed acquisition of competences and awareness by students; (2) on the other hand, to identify the specific characteristics of a teacher who is able to act in order to foster students' acquisition of the key-competences in the use of algebra as a thinking tool and their development of an awareness of the meaning of the activated processes. These distinguishing features, which can be placed in some fundamental behavioural categories of the cognitive apprenticeship, characterize the $M_{AE}AB$ construct.

The following table summarizes the characters of a teacher who poses him/herself as a $M_{AE}AB$, relating them to our theoretical framework of reference: in the first columns the main roles of a $M_{AE}AB$ are recapped; in the other columns the associated teacher's actions (second column) and the reference to our theoretical framework (third column) are highlighted.

Roles played by a teacher who acts as a M-$_{AE}$AB	**Corresponding actions of the teacher**	**Reference our theoretical framework**
(a) *Investigating subject* and *constituent part of the class* in the research work being activated	He/she tries to stimulate in his/her students and attitude of research towards the problem being studied	It is in tune with *Vygotsky*'s ideas of learning as a social process, according to which the interaction with adults or with more expert peers enables students activate internal learning processes which help them achieve a higher level of mental development.
(b) *Practical/ Strategic guide*	He/she shares (rather than transmit) with his/her students the adopted strategies and the knowledge to be locally activated.	It refers to the *modeling category of cognitive apprenticeship*: it requires that an expert performs a task externalizing the internal processes in order to make students observe and build a conceptual model of the processes that are required to accomplish it.
(c) *"Activator" of processes of generalization, modelling, interpretation and anticipation*	He/she provokes and stimulates the construction of the key-competences for the development of thought processes by means of algebraic language	The teacher has to play the role of both: - activator of *interpretative processes* (fostering a correct identification of the *conceptual frames* that have to be chosen to correctly interpret and transform algebraic expressions and a good *coordination between different frames*) and - activator of correct *anticipating thoughts*. At the same time it refers to the *categories* of *modeling*, *coaching* (which consists of offering to students hints and feedback while they carry out a task) and *scaffolding* (it refers to the supports the teacher provides to help students carry out a task).
(d) *Guide in fostering a harmonized balance between the syntactical and the semantic level*	He/She helps his/her students control the meaning and the syntactical correctness of the algebraic expressions they construct and, at the same time, the reasons underlying the correctness of the transformations they perform.	It requires to foster a good coordination between the verbal and the algebraic *register*, through the activation of *correct conversions* and *treatments* (Duval 2006). This role also refers to the *articulation category* of cognitive apprenticeship, which involves the methods applied to make students articulate their knowledge, way of reasoning and problem-solving processes.

(e) *Reflective guide*	He/She stimulates reflections on the effective approaches carried out during class activities in order to make students identify effective practical/strategic models from which they can drawn their inspiration.	Through this role, the teacher highlights those processes which can be associated to an effective activation of the *three key-competences* in the use of algebraic language as a thinking tool. It refers, in particular, to the *reflection category* of cognitive apprenticeship, which involves enabling students to compare their own problem-solving processes with those of an expert or of another student, so that they ultimately could be able to compare them with an *internal cognitive model of expertise*.
(f) *"Activator" of both reflective attitudes and meta-cognitive acts*	He/She stimulates and provokes meta-level attitudes, with a particular focus on the control of the global sense of processes.	The focus is on the *control of the processes associated to a real acquisition of the key-competences* in the use of algebraic language as a thinking tool. Again the reflective practices that the teacher aims at provoking are focused on the *articulation* of the activated processes in order to evaluate their appropriateness. Playing this role, which also refers to the *reflection category*, fosters, in tune with the ideas developed by *Leont'ev*, students' development of a real awareness of the meaning of both the class activities and the learning processes themselves.

The first three roles (a, b, c) that a teacher should perform in the class require him/her to carry out the activities posing him/herself not as a "mere expert" who proposes effective approaches, but as a *learner* who faces problems with the main aim of making the hidden thinking visible, highlighting the objectives, the meaning of the strategies and the interpretation of results.

The other three distinctive characteristics of the profile of a teacher as a $M_{AE}AB$ (d, e, f) refer to a different role played by the teacher: he/she must also be a point of reference for students to help them clarify salient aspects at different levels, with an explicit connection to the knowledge they have already developed.

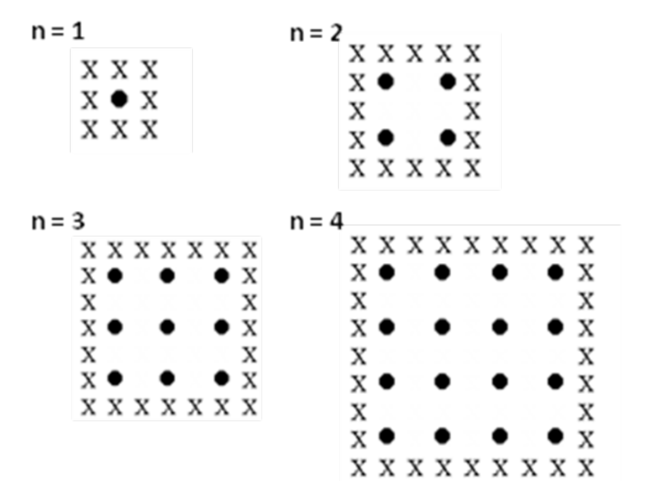
The previous table highlights that the algebraic and the social/methodological dimensions of our theoretical framework result to be complementary in combining to each other to enable the identification of the distinctive elements of an approach aimed at fostering students' aware and effective use of algebraic language as a thinking tool. In this way the $M_{AE}AB$ construct could be conceived as a tool which can favour an analysis of teaching practices that goes beyond the discourse used by the teacher in such a manner as to highlight the underlying intentions of the teacher, intentions that link directly to the mathematics at stake.

For this reason, it is important to stress that our aim is not to give an exhaustive definition of "effective teaching" as an absolute. Nevertheless, through the distinctive features of the $M_{AE}AB$ we aim at identifying those teaching practices that can directly influence the quality of student learning in the specific context of the teaching of algebra. At the same time, the construct enables the identification of those attitudes and behaviour that can negatively influence students' learning. As we stated before, in fact, our studies allowed us to contrast the positive effects of this kind of approach with the effects of an approach which is not in tune with the $M_{AE}AB$

construct, which can provoke students' development of a sort of pseudo-structural approach to the use of algebraic language as a thinking tool (Cusi 2012).

3. THE ANALYSIS OF A CASE

In this paragraph we analyze an excerpt of a class discussion focused on an introductory activity to algebraic modeling, referring to the $M_{AE}AB$ construct as an a-priori tool for analyzing teaching.
The excerpt refers to the initial part of a discussion conducted in a first class of lower secondary school (grade 6). The teacher, who is really motivated and interested in engaging with projects aimed at fostering curricular innovation, has collaborated with us in different experimentations, showing to have deeply grasped the aims, the methodology and the meaning of the approach subtended to the activities we promote. Since the following discussion constitutes a good example of conduction of class activities aimed at the introduction of algebraic modelling, we chose to focus on this excerpt in order to test the effectiveness of the $M_{AE}AB$ construct in providing "transparent" indicators to describe effective teachers' interventions.
The specific activity was proposed at the end of an introductory path to the algebraic modelling of figural sequences. The problem situation was adapted from the Pisa task usually named as "the apple trees". The characterizing feature of this task is the combination of the figural and verbal registers with the aim of fostering generalization and the algebraic formalization of the relationship between the number of apple trees and the number of conifers in the different possible configurations. In order to simplify the problem situation and to help students in its exploration and in making the identified relationships explicit, tables were introduced together with the requirement of specific argumentations. Due to space limitations we do not present the original worksheet, but only the proposed patterns and the first questions.

	Below you can find the patterns which represent the disposition of apple trees and conifers in relation to the number (n) of the rows of apple trees. 1) After having carefully observed the patterns, what can you say about the disposition of apple trees and conifers in the different cases? 2) Try to reproduce, through a drawing, the disposition of apple trees and conifers when n=5. Motivate your answer. 3) Explain how you can find the number of apple trees if you know the number of rows.

In the class discussion that we propose, the teacher tries to guide her students to the exploration of the number of conifers and apple trees in the different patterns.
The left column of the following table contains the excerpt of the first part of the discussion (T stands for the teacher, while the other alphabetical letters stand for the different students who take part in the discussion; due to space limitations we will skip some interventions which are not fundamental in the development of the

discussion). In the right column we propose an analysis of the teacher's interventions with reference to the $M_{AE}AB$ construct.

<table>
<tr><th>Class discussion excerpt</th><th>Analysis of T' interventions through the $M_{AE}AB$ construct</th></tr>
<tr><td colspan="2">The class exploration starts with T's request of reproducing the patterns on the workbook while she is doing the same at the blackboard.</td></tr>
<tr><td>1. T: What did you check, while I was drawing on the blackboard, to exactly reproduce the disposition of apple trees and conifers?</td><td>T poses herself as an investigating subject, stimulating an attitude of research towards the problem. Moreover she simulates an attitude of sharing.</td></tr>
<tr><td colspan="2">2. M: (I checked) how many conifers there are on each side.
... Other pupils intervene.
6. K: In the first drawing there are 9 conifers.</td></tr>
<tr><td>7. T : In the first drawing there are 9 conifers. How did you determine the correct number of conifers, K?</td><td>T poses herself as a reflective guide. When K looks at the total number of the conifers and makes a mistake, T does not express any judgment. On the contrary, she intervenes to turn K's attention to the counting strategies he adopted in order to prompt a correct attitude of inquiry and to foster a self-correction.</td></tr>
<tr><td colspan="2">8. K: I did 3... 3… I got wrong.</td></tr>
<tr><td>9. T: Try to explain that.</td><td>T poses herself as an activator of metacognitive acts: she fosters an attitude of enquiry, encouraging K so that he can be able to make his thoughts explicit.</td></tr>
<tr><td colspan="2">10. K: They are 8. I considered 3 at the beginning, on the first side, then I added 2, then 2 on the other side, and then 1.</td></tr>
<tr><td>11. T: Eight. Good, K! And how many apple trees are there instead, G?</td><td>T encourages again the students and poses herself as a practical-strategic guide, making them focus on the first configuration and re-directing the inquiry towards the identification of the interrelation between the number of conifers and the number of apple trees.</td></tr>
<tr><td colspan="2">12. G: One!</td></tr>
<tr><td>13. T: Let's explore the other representations as well. How many conifers are there in the second representation?</td><td>T poses herself as a participant, constituent part of the class group, and as a strategic guide, drawing students attention toward the second configuration.</td></tr>
<tr><td colspan="2">14. GF: 8 multiplied by 2. Two is the number of the rows. Therefore 16.</td></tr>
<tr><td>15. T: A said that he would have wanted to know how many apple trees are exactly in the drawing.</td><td>T poses herself as an activator of reflective attitudes, trying to focus students' attention to a comparison between the different cases. Focusing on the objective of the discussion, she is also trying to activate correct anticipations, with the aim of making them highlight a correlation between the number of conifers and the number of apple trees.</td></tr>
<tr><td colspan="2">16. GP: In this one there are 4 (apple trees)
17. M: I noticed that the number of rows is equal to the number of apple trees in the rows.</td></tr>
<tr><td>18. T: What would you say about M's</td><td>T does not judge M's observation and ask the other</td></tr>
</table>

<table>
<tr><td>observation?</td><td>students to examine it, posing herself as a reflective guide, with the aim of both stimulating reflections on the different approaches proposed and making them explicit.</td></tr>
<tr><td colspan="2">19. A: It’s right. When n=2 there are two apple trees in every row.
20. G: So, in order to calculate the number of apple trees in the enclosure we should multiply the number of the rows by the number of trees in every row.
21. K: I didn’t understand anything.</td></tr>
<tr><td>22. T: The observations actually overlapped.</td><td>When K declares his doubts, T poses herself again as a participant, stimulating the class in order that the different proposed observations could be better made explicit. In this way she fosters the sharing of knowledge and poses herself as an activator of both reflective attitudes and metacognitive acts.</td></tr>
<tr><td colspan="2">23. G: I meant to say that in this case, K, in order to calculate the number of apple trees you must take the number of apple trees in every row and multiply it by the number of rows. Therefore two multiplied by two.
24. M: That is you must multiply the number of rows by itself because the number of apple trees is equal to the number of rows.</td></tr>
<tr><td>25. T: So let’s see if I am able to understand. What do “the number of rows” and “the number of apple trees in every row” mean?</td><td>Instead of evaluating G and M’s observations, T poses herself as a reflective guide, asking students’ to clarify the meaning of some terms, with the aim of “making their thinking visible”. At the same time she poses herself as an activator of interpretative processes, trying to stimulate correct conversions from the verbal to the symbolic register.</td></tr>
<tr><td colspan="2">26. A: The rows are those (he points at the drawing) ... that is the number of rows, how many rows there are. The number of apple trees is how many apple trees there are in every row.
27. K: I have understood!</td></tr>
<tr><td>28. T: I have understood now. Thanks, M. So how can we write this number 4 which stands for the number of apple trees?</td><td>T stimulates and provokes the construction of key-competences for the development of thought processes by means of algebraic language, posing herself as an activator of interpretative processes, gradually stimulating the activation of correct conversions from the verbal to the symbolic register.</td></tr>
<tr><td colspan="2">29. Group of students: 2 multiplied by 2!</td></tr>
<tr><td colspan="2">The discussion continues with the analysis of the number of the conifers in every pattern and the following identification of the symbolic expressions which represent the relation between the number of apple trees and the number of conifers in every configuration and the number of rows. Lastly it ends with a naive study of an inequality in order to determine in what cases the number of the apple trees exceeds the number of the conifers.</td></tr>
</table>

4. FINAL REMARKS

The analysis we conducted testifies that the $M_{AE}AB$ could represent an effective diagnostical tool in the analysis of the quality of the teacher’s management of introductory activities to algebraic modeling.Through the theoretical lenses we adopted, in fact, it was possible to highlight an effective action of the teacher, characterised by a specific focus on the strategies aimed at making students control their thinking processes and develop an awareness about the meaning of the performed activities. In this discussion the teacher's interventions associated to the roles strictly connected to the algebraic dimension of our construct are less frequent than the roles associated to the meta-cognitive dimension. For this reason, the construct could also enable to highlight that, while during global-meta-level activities (Kieran 1996) there is a need of a good balance between the algebraic dimension and the meta-cognitive dimension, introductory activities to algebra requires a major focus on those roles which can better help students develop a deep awareness of the meaning of the processes they are involved in.

We believe that the $M_{AE}AB$ construct could be also a useful tool to promote teachers' reflection on their own practice. In tune with Mason's idea of teaching as "educating awareness" (1998), we think that making the teachers analyse their class processes through specific theoretical lenses could provoke what Mason defines "shifts of attention", which play an essential role in fostering the development of new awareness and hence in determining an effective teaching. We believe indeed that these activities could allow teachers to perform their first "guided" reflective practices, receiving and afterwards interiorizing the necessary stimulus for the construction of their own models for reflection, to which they can refer everytime they have to analyse their practice. In the future we intend to test this hypothesis referring to the $M_{AE}AB$ construct in the work with both pre-service and in-service teachers, proposing it to them as a tool for self-analysis.

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