TEACHING MULTIDIGIT MULTIPLICATION: COMBINING MULTIPLE FRAMEWORKS TO ANALYSE A CLASS EPISODE

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Abstract: This paper provides an analysis of a teaching episode of the multidigit multiplication algorithm, with a focus on the influence of teacher's mathematical knowledge on his teaching. The theoretical framework uses Mathematical Knowledge for Teaching, mathematical pertinence of the teacher and structuration of the milieu in a downward and upward a priori analysis and an a posteriori analysis. This analysis shows a development of different didactical situations and some links between mathematical knowledge and pertinence. In the conclusion, the contribution of the two traditions originated frameworks is briefly addressed.

This paper originated in a doctoral research project (Clivaz, 2011) that aimed to describe the influence of the mathematical knowledge of primary school teachers on their management of school mathematical tasks. The origin of that question partly came from U.S. mathematics education research and partly from the French *didactique des mathématiques*. Ball's categories of Mathematical Knowledge for Teaching (Ball, Thames, & Phelps, 2008) were used to describe the teacher's knowledge, while I described the effect on teaching through mathematical pertinence of the teacher (Bloch, 2009). I analysed the teacher's knowledge and the effect of this knowledge in ordinary classroom situations with the model of structuration of the milieu (Margolinas, Coulange, & Bessot, 2005) to take into account the complexity of the teacher's activity.

After a brief explanation of these three frameworks, their interaction will be shown through an episode about the teaching of the algorithm of multidigit multiplication. Finally, I will discuss the interaction of these frameworks for analysing the teacher's knowledge and teaching.

FRAMEWORK

CATEGORIES OF MATHEMATICAL KNOWLEDGE FOR TEACHING

Refining Shulman's (1986, 1987) categories of teacher knowledge for mathematics, Ball, Thames, and Phelps (2008) provide a practice-based division of Mathematical Knowledge for Teaching (MKT) (Figure 1).

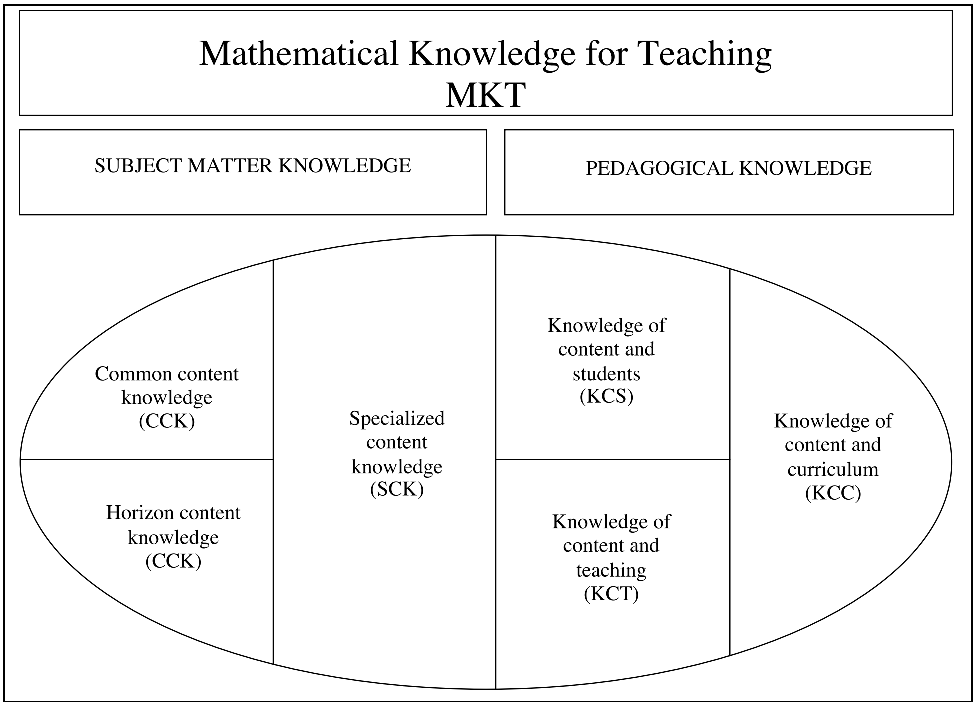

Figure 1: Domains of mathematical knowledge for teaching (Ball et al., 2008, p. 403)

One of the special features of this categorization is the existence of a *Specialized Content Knowledge,* defined as

> the mathematical knowledge and skill unique to teaching. In looking for patterns in student errors or in sizing up whether a nonstandard approach would work in general, […] teachers have to do a kind of mathematical work that others do not. […] This work involves an uncanny kind of unpacking of mathematics that is not needed–or even desirable–in settings other than teaching" (Ball et al., 2008, p. 400).

One of the examples of the use of MKT Ball and her colleagues often provide is the teaching of the multiplication of whole numbers algorithm (Ball, Hill, & Bass, 2005, pp. 17-21). I studied examples of this teaching with four teachers in the Lausanne region of Switzerland (Clivaz, 2011), and the case I will analyse is about this teaching.

MATHEMATICAL PERTINENCE OF TEACHER'S ACTIONS

In order to detect the effects of a teacher's mathematical knowledge, Bloch (2009) suggests considering the *mathematical pertinence of teacher's actions*. An action is pertinent if it allows the student to grasp the functionality of mathematical object, with enouncement of mathematical properties, mathematical arguments for the validity of procedures, or for the nature of mathematical objects[1]. Bloch gives three criteria for this pertinence, the first being the "ability to interact with the students on mathematical aspects of the situation and to encourage their activity by the means of interventions and feedback on their mathematical production[2]" (p. 32).

STRUCTURE OF THE MILIEU

To describe the teacher's activity, Margolinas (Margolinas, 2002; Margolinas et al., 2005) has developed a model of the teacher's *milieu*[3], based on Brousseau (1997), which she also uses as a model of the teacher's activity.

+3	*Values and conceptions about learning and teaching* Educational project: educational values, conceptions of learning, conceptions of teaching
+2	*The global didactic project* The global didactic project, of which the planned sequence of lessons is a part: notions to study and knowledge to acquire
+1	*The local didactic project* The specific didactic project in the planned sequence of lessons: objectives, organization of work
0	*Didactic action* Interactions with pupils, decisions during action
-1	*Observation of pupils' activity* Perception of pupils activity, regulation of pupils' work

Figure 2: Levels of a teacher's activity (Margolinas et al., 2005, p. 207)

At every level, the teacher has to deal not only with the current level, but at least also with the levels directly before and after the current level. This tension makes a linear interpretation of teacher's work inaccurate (Margolinas et al., 2005, p. 208). In fact, a more complete model can be considered, including the student (E, for *élève*), the teacher (P, for *professeur*), and the milieu (M). Each milieu M_i is constituted at each level i by the lower E_{i-1}, $P_{i-1,}$ and M_{i-1} component and the situation S_i is made at each level i by E_i, $P_{i,}$ and M_i. This can be written as $S_i = (M_i ; E_i ; P_i)$ and $M_i = S_{i-1}$, or more visually represented in an onion diagram (Figure 3) or in a table where the teacher's levels range from +3 to -1 and the student's levels range from -3 to +1.

Therefore, S_0, the didactical situation, can be determined either from the teacher's point of view, by a *downward analysis*, or from the student's perspective, by an *upward analysis*. This latter may conduct to one or more didactical situations S_0 which may not be the same as the situation determined by the downward analysis. Margolinas (2004) calls this a *didactic bifurcation*.

The downward analysis uses mainly the audio-recorded interview we had with each teacher before the lessons about multidigit multiplication. The upward analysis is *a priori* "in the sense that it doesn't depend on experimental or observational facts[4]" (Margolinas, 1994, p. 30). Both are then tools to analyse the classroom observations in an *a posteriori* analysis.

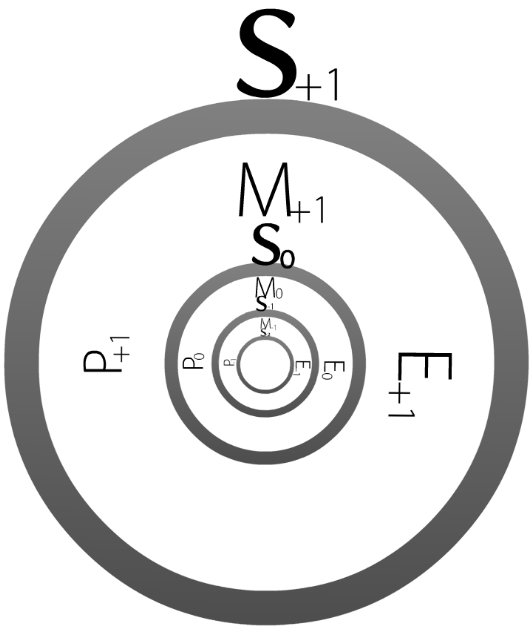

Figure 3: Structure of the milieu, level -1 to +1

THE CASE OF DOMINIQUE

The episode, 27 minutes in duration, that was analysed is taken from the series of seven lessons Dominique, a Grade 4[5] teacher, devoted to this algorithm in his class. The analysed episode features the moment where Dominique plans to show the multidigit multiplication algorithm. The video recording was transcribed and coded with Transana software (Fassnacht & Woods, 2002-2011) according to the categories of mathematical knowledge for teaching, mathematical pertinence, and the levels of the teacher's activity. This extract can be situated in the series by means of a *synopsis* (Schneuwly, Dolz, & Ronveaux, 2006) and a *macrostructure* (Dolz & Toulou, 2008).

The *a priori* downward analysis will be presented in the following sections of this paper. Due to space constraints, the *a priori* upward and *a posteriori* analyses cannot be developed here. We will very briefly describe the situations S_0 these *a priori* analyses reveal and, in the *a posteriori* analysis, we will suggest reasons for some highlighted troubles, in terms of types of Mathematical Knowledge for Teaching (MKT) and mathematical pertinence.

A PRIORI ANALYSIS

Downward analysis

Based on the interview with the teacher, the downward analysis goes from level +3 to level -1 and determines *a priori* the didactical situation S_0 from the teacher's point of view. The topics Dominique addresses are various, so I focus only on the question about the type of algorithm, the MKT linked to that question[6], and the consequence about the determination of S_0 didactical situation.

Dominique thinks that pupils should understand what they do in math (level +3). He also feels this way about the algorithms, but he views algorithms as tools for problem solving (+3). So, for the series of lessons about multidigit multiplication, the main goal is that the students are able to carry out the algorithm and use it efficiently (+2). The type of algorithm is not important if it is efficient for the students (+2). Dominique knows that there are several kinds of algorithms for multidigit multiplication, and he plans to show two of them: the "table algorithm" (Figure 4, left) and the algorithm *en colonnes*[7] (EC) (Figure 4, right). This way of showing more than one algorithm is consistent with the official regional instructions (DFJ, 2006) and textbooks (+3) (Danalet, Dumas, Studer, & Villars-Kneubühler, 1999). At the end of the chapter of the textbook about this topic, Dominique will ask students to only retain and use the EC algorithm, with the justification that this is the algorithm everybody learned at school, and it is more efficient than the table algorithm (+2).

For the lesson, Dominique plans to show first the table algorithm on a two-digit by two-digit multiplication question and then to show the EC algorithm on the same example. He knows that the two algorithms give the same results and that the partial results can be compared line by line (+2), and he plans to show that (+1). However, he doesn't mention any other link between the table algorithm and EC – in general (+2), when planning the lesson (+1), or when envisioning teaching the lesson (+1). In addition, he does not observe the students using the two algorithms on the same multiplication question (-1). To make the line-by-line comparison possible, he plans on asking the students to "put the tens below": "It's not very logical, but it allows the student to have the two (lines) in front of each other"[8]. He never mentions any other reason or justification for this step and never mentions the possibility (and the effects) on the inversion of the two factors (+2).

Dominique thinks that the only problems pupils will face in the EC algorithm are multiplication facts and the second line zero (+2). He foresees that he will observe many errors about this zero (-1). So he plans to repeat the *zero rule*: "when one works with tens, a zero must be added"[9].

For Dominique, multiplication is shortcut for addition (+2). He never plans to mention any link with area when explaining the table or EC algorithm (+1) even if he asked one area problem to introduce the necessity for building an algorithm (+2). Challenged about his representation of multiplication, he never gives any other representation, and when asked about the link between multiplication and area, he answers that the area has to be computed with multiplication (+2).

These elements contribute to determine the didactical situation S_0 from the teacher's perspective. It can be summarized in four points:

1. Show the table algorithm for the example 12×17, requiring writing the units first for 17.
2. Show the EC algorithm on the example 12×17. Write the EC algorithm next to the table. After the first line, highlight the fact that the results of both algorithms' first lines are the same.
3. Write the zero at the right place in the second line, because "when one works with tens, a zero must be added". Carry out the second line and highlight the fact that the results of both algorithms' second lines are the same.
4. Finish by adding the two lines.

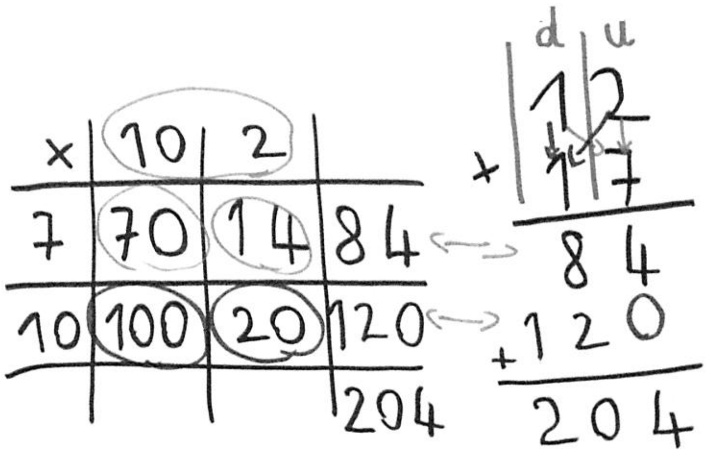

Figure 4: Two algorithms for the multiplication 12x17, written by Dominique on a poster.

Upward analysis

The elaborate upward analysis (Clivaz, in press) starting from M_{-3} material milieu, shows that the student can deal with M_{-3} in different ways about the parallelism between the table and the EC: one row with one row (same correspondence as the teacher), partial product to partial product, just copy the results or do the two algorithms independently. He/she can also apply the *zero rule* in three ways: he/she can write a zero before beginning the second row with no further interrogation, he/she can link this zero to each zero in the table's second row, and he/she can literally apply the teacher's explanation, adding a zero each time he/she works with a ten. The combination of these two dimensions conducts to twelve situations S_{-2}, from which six seem consistent and lead to six S_0 didactical situations.

A POSTERIORI ANALYSIS

The detailed *a posteriori* analysis of the 27-minute episode (Clivaz, 2011, pp. 194-204; in press) compares the *a priori* analysis with the video and the transcript of the actual lesson. It shows that most of the students considered the two algorithms independently or in a line-by-line correspondence, and wrote the zero without interrogating. These students were in two of the S_0 situations that the upward analysis determined. However, one student, Armand, repeatedly asked questions about why the teacher didn't add a zero *each time* he used tens. He was in another S_0 situation. He also asked several times, "Is it 1×1 or 10×10 ?[10]". However Dominique kept *his* S_0 situation and was not able to even understand Armand's interrogations.

MKT and pertinence

The proliferation of didactical bifurcations and the inability of the teacher to notice that the Armand's S_0 radically differed from his have their origin in the teacher's choices made at the +3 to 0 levels. Additionally, these choices may be understood as a consequence of the teacher's MKT.

The first choice was to use the table algorithm and particularly the correspondence between the lines' sum in the table and in the EC's lines, but with no explicit correspondence between each partial product. Dominique's MKT about the table were accurate, as revealed in the interview, but they were not pertinent, since they didn't allow him to interact with the students on the mathematical parts of the situation (pertinence's first criteria according to Bloch, 2009). The reason for this discrepancy between knowledge and pertinence was the knowledge of multiplication itself. For Dominique, multiplication was only a shortcut for repeated addition. He never considered it as a Cartesian product or as the area of a rectangle. Therefore, to Dominique, EC and table algorithm were two ways to perform multiplication; they were not linked to multiplication itself and they were just linked to each other because they gave the same result.

The second choice was the "recipe" for *zero rule*. This rule is problematic in many ways: use of additive words (add a zero), lack of link with place value, and above all, fallacy if literally applied. Regarding these two choices, Dominique had a working *Common Content Knowledge*, but he couldn't unpack it and couldn't use the corresponding *Specialized Content Knowledge* to explain why a zero appears when one multiply by tens.

CONCLUSION

This episode analysis used the structure of the milieu to highlight and to analyse links between mathematical knowledge for teaching (MKT), pertinence, and teaching choices of the teacher. It showed that not only Common Content Knowledge is necessary to apply pedagogical MKT, but also "that each of these common tasks of teaching involves *mathematical* reasoning as much as it does pedagogical thinking" (Ball et al., 2005, p. 21). It is one illustration of the way one U.S. mathematics education framework and elements of the *Theory of didactical situations* (Brousseau, 1997) can interact to analyse a math teaching issue. The original question was about mathematical knowledge of the teacher, but the finesse of the structuration of the milieu was necessary to show the multiplicity of the various didactical situations. The distinction of specialized content knowledge among MKT was needed to analyse the causes of the phenomena when the structuration of the milieu and the pertinence were crucial to capture the movement of didactical situations beyond the static character (Ball et al., 2008, p. 403) of Ball's categories.

The combination, in the sense of Prediger, Bikner-Ahsbahs, and Arzarello (2008), of the frameworks from two cultural backgrounds allowed to "get a multi-faceted insight into the empirical phenomenon in view" (p. 173). It was more vastly used in other parts of the doctoral research (Clivaz, 2011) and gave some interesting results, for example about the correlation between MKT and pertinence. For that purpose, we invite the interested reader to read the full dissertation.

These two frameworks also raise one more general issue. The question of mathematical knowledge of the teacher is widely studied in the English-speaking mathematics education community, but, according to the National Mathematics Advisory Panel (2008), it was often studied with the quantitative point of view of the influence on students' test outcomes. It is far less disputed[11] in the French speaking *didactique des mathématiques*, even though the developed model could offer tools to discuss the influence of MKT on teaching. I hope that studies using frameworks originated in the two contexts will be developed to connect different theoretical approaches as promoted in the ERME spirit.

NOTES

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1 « Une intervention mathématique est pertinente si elle rend compte dans une certaine mesure de la fonctionnalité de l'objet mathématique visé ; ou, s'agissant d'enseignement, si elle permet au moins de progresser dans l'appréhension de cette fonctionnalité, avec des énoncés de propriétés mathématiques contextualisées ou non, des arguments appropriés sur la validité de procédures ou sur la nature des objets mathématiques. » (Bloch, 2009, p. 32)

2 « […] capacité à interagir avec les élèves sur des éléments mathématiques de la situation et à encourager l'activité des élèves par des interventions et des retours sur leur production mathématique. » (Bloch, 2009, p. 33)

3 Milieu is the usual translation for Brousseau's French term "milieu", but, in French, it refers not only to the sociological milieu but it is also used in biology or in Piaget's work. A more accurate translation would be "environment".

4 « dans le sens qu'elle ne dépend pas des faits d'expérience ou d'observation », my translation.

5 9-10 year-old students.

6 The type of MKT is not mentioned in this short version of the *a priori* analyses.

7 Literally "in columns" but the accurate English name would be "long multiplication".

8 « Mettre les dizaines en dessous. C'est pas très logique, mais ça permet d'avoir les deux en face. »

9 « Quand on travaille avec les dizaines, on ajoute un zéro. »

10 « c'est 1×1 ou 10×10 ? »

11 With the notable exception of Quebec and the presence of a Working Group on the topic in EMF congress (Dorier & Sylvia, 2012).

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