DIDACTICAL ANALYSIS AND COMPARISON OF TEACHERS' PRACTICES

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We report the use of an analytical tool to analyze teaching and learning processes and display essential elements of mathematical activity (definitions, propositions, properties, mathematical processes, etc.) during the development of a class. It has been applied to the study of the commonalities and differences among three classes conducted by three different teachers in the same institution, year and school level when they teach about the perpendicular bisector. The results allow us to infer some aspects about the mathematical knowledge of the teachers involved.

INTRODUCTION

The research on mathematical knowledge and the professional development of teachers has become increasingly important in recent years, and has revealed not only its complexity but also the limitations of the results (Sullivan and Wood, 2008). Our work deals with the mathematical knowledge of teachers observed in their teaching practice. Like many other authors, we emphasize how the mathematical content knowledge of teachers is manifested in their classes as a good practice (Rowland, Huckstep and Thwaites, 2005; Fernandez and Yoshida, 2004; Davis, 2008). From this standpoint, the goal is to improve mathematical practices in the classroom while focusing on both, the *a priori* complexity of mathematical objects, and the teacher's mathematical knowledge when they use those mathematical objects. This approach produces research whose aim is to account for the actions that allow teachers to develop their profession successfully (Mason, 2002). Other authors emphasize the importance of being competent in didactical analysis, which might allow the teacher firstly to identify and organize the multiple meanings of the concept to be taught, and secondly, to select some processes for instruction and learning. To carry out such kind of didactical analysis is necessary to design tools which permit to account for the complexity of mathematics education. Some methodological frameworks have been developed in this direction (Godino, Contreras and Font, 2006).

We assume that the complex nature of mathematical objects may be perceived for example in different institutions and different historical moments, different textbooks, or different methodological approaches in the classroom. From this stand point follows that comparing the practice of different teachers working in the same institution, presenting the same mathematical object at the same level and at the same time, will enrich our understanding of a mathematical content knowledge in practice. From this standpoint, the objectives of this research are as follows:

- To design a visualization tool for a math class that may account for its complexity in terms of mathematical objects and processes, and their interrelationships.

- To use this tool as an analytical tool for teaching which: a) highlights the essential elements of mathematical activity during the course of a class, and b) accounts for some aspects of mathematical knowledge used by the teachers during their practice.

- To account for the commonalities and differences of the mathematical activity of different teachers presenting the same mathematical content in the same school year and the same institution.

As we would like to emphasize the methodological features and outcomes of our contribution, the theoretical framework, specially concerning the way in which the onto-semiotic approach is used, is embedded in the next two section. We report the how we conducted the analysis of the three classes and a detailed analysis of one of them in terms of primary objects and mathematical processes, as detailed below.

MATHEMATICAL ACTIVITY, OBJECTS AND PROCESSES

We observed the practice of three different teachers (hereinafter Laura, Antonia and Encarna) when they taught the perpendicular bisector in the final year of primary school (aged 11-12). They carried out different mathematical activities showing different models of classroom management. The three classes were videotaped and transcribed for later analysis, looking for commonalities and differences in the mathematical activity.

For the analysis of transcripts we used the first two of the five levels of the didactical analysis model proposed by the ontosemiotic approach (Godino, Batanero and Font, 2007; Godino, Contreras and Font, 2006; Pochulu and Font, 2011). The first one explores mathematical practices in mathematics instruction processes, defined as sequences of actions subject to mathematical rules. In this case, all three teachers share a common practice, namely constructing the perpendicular bisector. The second focuses on the primary objects and mathematical processes involved in conducting practices as well as those emerging from them. The three remaining levels are oriented respectively to find interaction patterns, norms, and didactical suitability of the practice. For the ontosemiotic approach (hereinafter OSA), mathematical activity plays a central role, and is modelled in practices where primary objects emerge. OSA use “primary object” in a wide sense to mean any entity which is involved in some way in mathematical practice or activity and to which we can refer uniquely. We will refer here to definitions, properties, construction procedures and problems. On the other hand, instead of giving a general definition of process, OSA opts to select a list of processes that are considered important in mathematical activity, without claiming that such a list includes all the processes implicit in all mathematical activities. Tasks or problems are considered objects because they are triggers of the mathematical activity.

An example to distinguish practices, primary objects and processes is the construction of the perpendicular bisector. For this construction, the student performs a sequence of actions subject to mathematical rules (practice in EOS), such as those underlying the use of the ruler and compass. In particular, students can use a construction procedure (algorithm) using only ruler and triangle 45 degrees (it is a procedure, which is considered a primary object type in EOS). With the repetition with other similar exercises the student engages in a process of algorithmization.

Detailed analysis of primary objects and mathematical processes illustrates relevant aspects of the structure and development of each of the classes and permits us to distinguish many elements of the mathematical activity as well as establish relationships between them. Metaphorically, we can say that we get a snapshot of the class, which is also the tool that allows us the comparison among the three classes. Tables 1 and 2 summarize the objects and processes (and their codes) that have been taken into account for the analysis of the practice of the three teachers.

Table 1: Mathematical objects and processes emerging from the practice.

Mathematical objects	
Definition of perpendicular bisector: Teachers refer explicitly or implicitly to one definition of perpendicular bisector. It includes also definitions of related as segment or line.	
D_1:	Perpendicular line passing through the midpoint of the segment.
D_2:	Locus of all points equidistant from two given points.
D_2A:	Locus of all points equidistant from the ends of the given segment.
D_2B:	Line (boundary) which separates the plane into two regions, so that in a region all the points are nearer one of the two points than the other.
Properties: Any statement regarding the definition and the construction method of the perpendicular bisector, which can be true or false, but there is an attempt to justify it in class.	
P_1:	The point where the line intersects the segment is the midpoint.
P_2:	The obtained line is perpendicular to the given segment.
P_3:	Points on the boundary (D_2B) are aligned.
Construction procedure: Construction algorithm of the perpendicular bisector.	
Pr_1:	Euclid's procedure (Book I prop. X): given a *finite straight line*, describe an equilateral triangle on it (Prop. I) and bisect its angle (prop. IX).
Pr_2:	Perpendicular bisector as a locus: Given two points, find any other two points equidistant from them and connect these last two points with a line.
Pr_3:	Carpenter's procedure: Given a segment, measure its length, take its half and draw the perpendicular at the midpoint with the triangles 45 or 60 degrees, or the protractor.

Problem: tasks that incite mathematical activity, examples and counterexamples.	
EP:	Task based on paradigmatic examples
ENP:	Task based on non paradigmatic examples
CE:	Counterexamples

Mathematical processes		
Institutionalization: A definition, property or procedure is explicitly considered as valid, so from that moment on it is assumed to be known.		
Automation: Students are asked to repeat a certain procedure mechanical and individually.		
Communication: Oral or written statements on mathematical contents are expressed or understood. We explicitly exclude from this category mathematical arguments. Three subcategories have been included:		
EP: Teacher's lecture	DPA: Dialog among teacher and students	DA: Dialog among students
Argumentation: Existence of chains of mathematical arguments		
Modelling: At least one of the following phases of modelling (Blom, 2002) occurs: (a) starting point is a certain situation in the real world; (b) simplify, structuring and making the content precise; (c) objects, data, relations and conditions involved in it are translated into mathematics, and mathematical results derive, (d) retranslation into the real world.		

COMMONALITIES AND DIFFERENCES AMONG THE THREE CLASSES

Laura's class is a lecture-type. Encarna's one is problem-solving based, directed and not constructivist, and Antonia's class combines elements of both previous management models. The analysis that we have carried out in terms of primary mathematical objects and mathematical processes allows us to draw some conclusions about the type of mathematical activity that promotes each one of these management models and how is distributed along time, as well as having a comparative overview of the three models at the same time. Figures 1 to 3 show the radiography of the three classes.

A first analytical approach to data shows that the mathematical activity is not uniformly developed during each one of the classes, but in all three we find slots of time in which the density of processes, definitions, procedures, etc. is higher. We will refer to these time intervals as accumulation time. In all three cases, these periods of time occupy approximately from one quarter to one third of the total time. However, in the case of Laura there is an important accumulation phase during the first ten minutes, when most primary objects are presented. In the case of Antonia this phase begins in the minute ten approximately, while in Encarna's class (which is significantly longer) appears at the end (Figure 3).

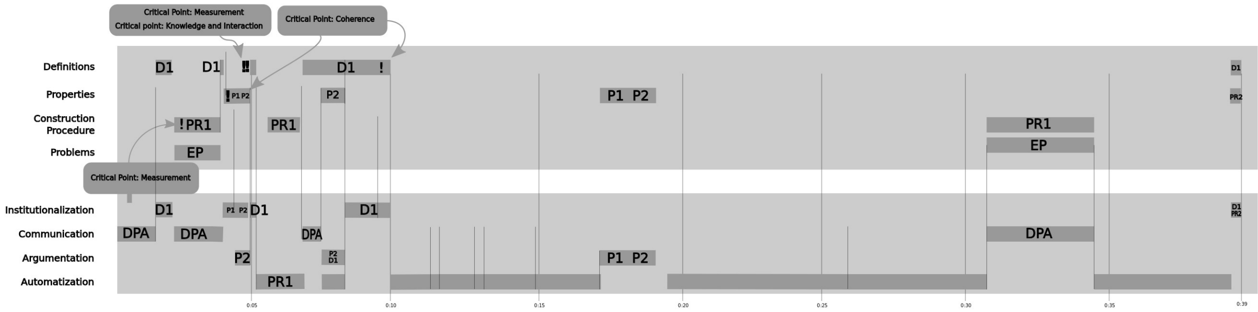

Figure 1. Radiography of Laura's class. See Table 1 for label descriptions.

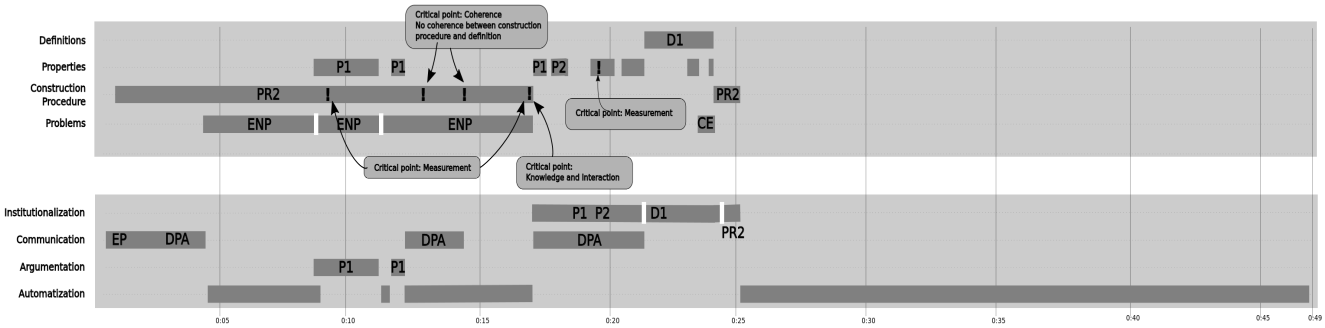

Figure 2. Radiography of Antonia's class. See Table 1 for label descriptions.

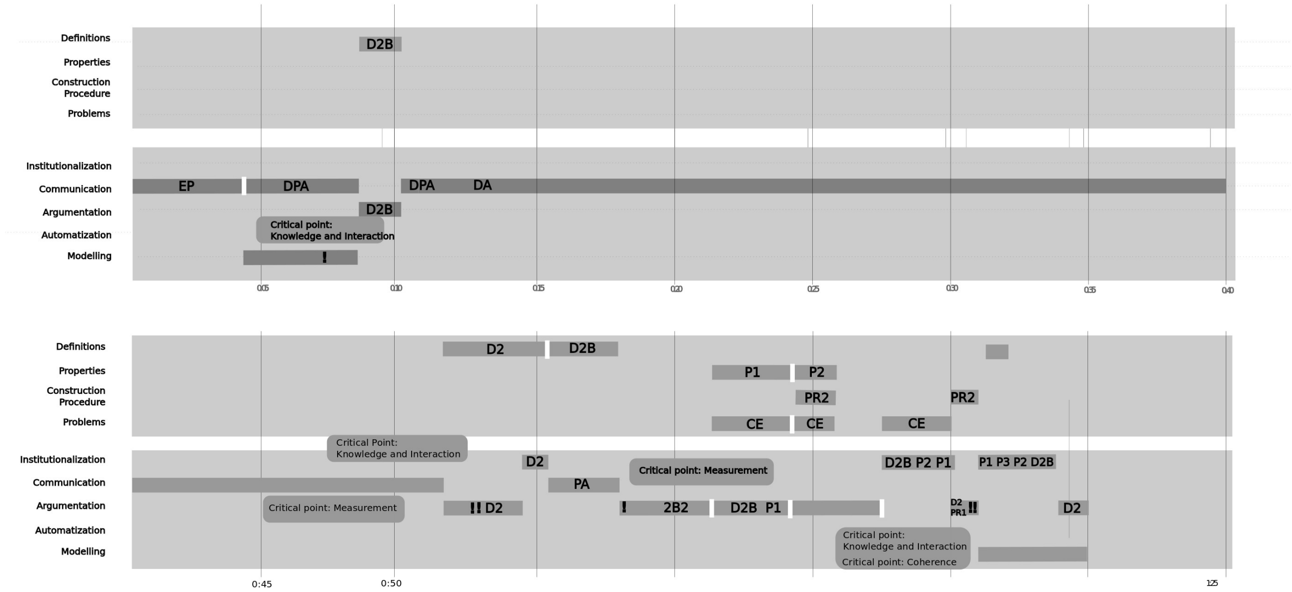

Figure 3. Radiography of Encarna's class (significatively longer). See Table 1 for label descriptions.

As a way of illustration, let us read the data displayed in Figure 1: Notice on the processess stripe that 75% of the time is consumed to automate one construction procedure. The predominant communication process is a lecture-type, and only a short time is devoted to the process of argumentation. Moreover, during the argumentation process there is a lack of logical consistency on the validity of using direct measurement for proving or produce geometrical constructions. Communication between students and teacher only occurs when the definition D1 and properties P1 and P2 are introduced. In all three cases there is a predominant process, which takes three quarters of the total time approximately, and appear different definitions and procedures which not always are properly institutionalized. In the cases of both, Antonia and Laura, it is the automation process, which starts immediately after the accumulation phase and is extended until the end of the class. In the case of Encarna's class, it is communication. We also notice that modelling and argumentation processes appear in Laura's class, while they have little or no presence in the other two classes.

DETECTION OF CRITICAL POINTS

The analysis of the moments of accumulation permit us to detect some critical points. A critical point is a manifestation of the difficulties or potential that the teacher has to deal with due to the complexity of the mathematical object (in this case the perpendicular bisector). Difficulties are manifested in the form of errors, omissions, inaccuracies, lack of logical consistency in the teacher's speech, etc. while the potential are manifested by actions positively oriented to improve student's learning. Seeking for those indicators in the transcription of the lessons, we have identified three critical points in teacher's knowledge about the perpendicular bisector. Two of them report on teachers' mathematical knowledge. These are the critical point labelled *coherence*, which is related to the lack of consistency between the process of constructing the bisector and the definition in use, and the critical point *measurement* which has to do with the teacher's difficulties when using direct measurement. The third critical point is labelled *knowledge and interaction* and informs about how the teachers use their knowledge depending on the requirements of the classroom. Critical points have been included in Figures 1 to 3 in rectangular boxes. Critical points related to direct measurement appear in the three classes and, in the following, we show some examples to develop some insights about them.

In Laura and Antonia's class, the critical point appears because of their difficulties to cope with the complexity of the perpendicular bisector. The two following extracts from Laura's class illustrate the lack of logical consistency of her discourse regarding the validity of the measure for proving or doing geometrical constructions. In the first one, a student intends to find the midpoint of the segment using a graduated ruler, but the teacher makes explicit that direct measurement is not permitted for the construction. However, in the second extract below, the same teacher uses the direct measurement with the angle protractor to verify that the properties of the definition hold.

Excerpt 1:

Teacher: So the bisector of the segment is nothing else than the straight line perpendicular to this segment that divides exactly it into two equal parts, right? What do you do to get the midpoint of that segment and split it into two equal halves? Say.

Student: I could put this on it -raising a ruler- and measure it.

Teacher: I could measure it with the rule but, would I obtain the same? Exactly, exactly?

Student: With the compass.

Teacher: [nodding] With the compass (takes the chalkboard compass). The compass is the right tool with which the midpoint of the segment is going to be perfect.

Excerpt 2:

Teacher: Therefore, one condition is that the line dividing the segment into two equal parts, the bisector of the segment, must be perpendicular. How can I know if these two lines are perpendicular? How do I have to do? Perpendicular (she points the four quadrants in the chalkboard)

Student: Measuring with the protractor.

Teacher: [nodding] Measuring with the protractor. (takes the chalkboard protractor)

Student: A right angle.

Teacher: and I have to obtain...

Students: A right angle, ninety.

Teacher: and I have to obtain four right angles. One, two, three, and four. If I put the protractor here (on the first quadrant)... Let's see. Note that I obtain exactly 90 degrees. OK? And If I put it this way I also obtain 90 degrees exactly. So I can say that the bisector of the segment is the line which is perpendicular to that segment and divides it into two perfectly equal parts. Exactly.

Investigating which are the critical points that emerge from teaching practices on a specific content is very important to connect the mathematical knowledge requiered to present this content with other mathematical knowledge that permits the teacher to manage students interventions and classroom activity properly. As we illustrated above using such critical points, it is necessary to have a robust knowledge about mathematical foundation knowledge concerning the role of proof, or the use of direct measurement when solving Euclidean geometry problems to deal with teaching practices like this about the perpendicular bisector.

FINAL REMARKS

The above described tool designed for visualizing the development of the class has permitted us to highlight commonalities and differences among the practice of different teachers about the same content, conducted in the same institution, and at the same moment. The tool highlight the emergence of objects and processes during the teaching activity and make evident that all of them show individually some lack of uniformity regarding the emergence of mathematical contents and processes. However, all three show some accumulation intervals of time where critical points emerge. These accumulation intervals are located at different times in each class, the teachers use different primary objects, and critical points are managed in a different way.

The analysis allows us to a) characterize the mathematical content that has been brought into play in the classroom, offering a detailed and systematic way of approaching the concept of perpendicular bisector from the teacher's point of view and, b) infer the limitations and potential of the knowledge of teachers to deal with the complexity of the perpendicular bisector. Such limitations and potential is firstly manifested through the differences in the design and implementation of teacher's classes and, secondly, on the way they deal with the critical points discussed. For example, in the case of Antonia, her higher level of mistakes and vagueness compared to the other two teachers, makes us infer that there could be a lack of basic knowledge about the bisector. However, in the case of Laura, we can infer that she understands the basic elements of the bisector but does not have a strong knowledge concerning foundation of mathematics, thus there is a lack of logical consistency concerning the use of direct measure for argumentation and proof. Moreover, this lack of consistency makes the students unable to distinguish properly what a mathematical proof is. Thus, their transition to secondary school, where argumentation and proof will be increasingly important, will be negatively affected. The reasons why the teachers give no importance to the rigor of proving has to do probably with the fact that primary school teachers do not make sense of mathematics as a science of proving, or know about the importance of this characteristic of mathematics for the students to learn mathematics in the future. In the case of Encarna, we infer a deeper understanding of the complexity of the specific mathematical content bisector, especially showed through the didactical approach she uses, and the way she manages the critical points.

Finally, our work intends also to contribute to the development of teachers' competence on didactical analysis during their initial or permanent education, thus we work on a) indicators to select rich classroom episodes, like the presence of critical points, and (b) the sistematic analysis of the mathematical knowledge that enables teachers to deal with complex teaching practices on concrete mathematical concepts.

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