NETWORKING THEORIES BY ITERATIVE UNPACKING

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An iterative unpacking strategy consists of a sequence of empirically-based theoretical developments so that at each step of theorizing one theory serves as an overarching conceptual framework, in which another theory, either existing or emerging, is embedded in order to elaborate on the chosen element(s) of the overarching theory. The strategy is presented in this paper by means of reflections on how it was used in several empirical studies and by means of a non-example. The article concludes with a discussion of affordances and limitations of the strategy.

INTRODUCTION

A long-term program of our research group focuses on identifying and characterizing learning that possibly occurs when individuals or groups of individuals are engaged in problem solving and problem posing. In several of our studies, a typical data set consists of a series of recorded observations of how a learner or a small group of learners approaches an insight problem or tries to openly generalize a known theorem or pose a mathematical problem, which would be interesting to solve also to the posers. Many events, interactions and developments occur in such situations, so we, as many other researcher groups, repeatedly face the following research choices:

- Which of the observed developments are worthwhile enough in order to call them "learning"?
- The learners' successes and failures with problem-solving and problem-posing tasks are functions of many conditions and variables, some of which are out of our reach. So, which variables and conditions should become a focus of our attention, if we wish not only to describe the learners' actions and "learning," but also to explain what stipulated them?

These questions presume different theoretically-laden answers in different circumstances. Though individual studies in our group heavily rely on selected theories, their elements or their combinations, we are not adhered, as a group, to a particular theory or conceptual framework[1]. Thus, the need to develop certain strategies of calling into play several theories in one study as well as in a sequence of studies emerged for us. The goal of this paper is to introduce one of such strategies, which I would like to term *networking by iterative unpacking*.

ITERATIVE UNPACKING STRATEGY: AN INTRODUCTORY EXAMPLE

Let me introduce an iterative unpacking strategy by an example taken from a recent article by Simon et al. (2010). Simon and his colleagues presented a novel approach to studying learning through mathematical activities. The approach was developed for capturing subtle processes of transition the learners come through when progressing from one conceptual step of knowledge construction to a subsequent one. The scholars contrasted their approach with an approach developed by Thompson (1994) and Steffe (2003). That approach includes identifying sequences of developmental steps in students' mathematical actions and analysing them, in particular, in Piagetian terms of perturbation, accommodation and reflective abstraction. To further situate their approach in the literature, Simon et al. asserted:

> Our work also builds on the work of Hershkowitz, Schwartz, and Dreyfus (2001) who took on the challenge of explicating the formation of abstractions. Toward this end they described three 'epistemic actions' in the process of abstraction: 'construction, recognition, and building with.' They emphasize that construction is the key part of the process. *Our work can be seen as attempting to unpack construction* [italics is added] (p. 80)

Notably, Simon et al. refer to the Hershkowitz, Schwartz and Dreyfus's (2001) work in a dual way. On one hand, they refer to it as a theory, which unpacks a particular element of a previously developed theory. To this end and consistently with the Bikner-Ahsbahs and Prediger's (2006) terminology, the former theory can be seen as a foreground one, and the latter – as a background one. On the other hand, they use the Hershkowitz et al.'s (2001) work as a background theory or as an overarching framework, in which their own foreground theory is embedded. Simon et al. specifically point out which theoretical construct of the overarching framework they are going to unpack. They then perform the unpacking by developing an elaborated kit of tools for analysing the process of construction/transition in terms of seeing commonalities between the tasks in a sequence of tasks and anticipating solutions.

My point is that Simon et al. utilized two networking strategies, one of which had been explicated in the literature (e.g., Bikner-Ahsbahs & Prediger, 2006; Prediger, Bikner-Ahsbahs & Arzarello, 2008) and another was not. First, they utilized a *comparison strategy* by pointing out the differences between their approach and the Piagetian-oriented approach developed by Steffe and Thompson. The main difference was that Simon et al. suggested exploring learning without necessarily focusing on perturbation. Second, though Simon et al. did not use the “construction, recognition, and building with” language in their analysis, they considered it as an overarching conceptual framework for their own theorizing. To this end their contribution is in unpacking one of the key constructs of the Hershkovitz et al.'s (2001) theory. Therefore – *unpacking strategy*. Furthermore, the whole process of theorizing presented in the Simon et al.'s (2010) article includes the following chain of iterations: (1) from focusing on perturbation, accommodation and reflective abstraction to unpacking the formation of abstraction in terms of construction, recognition, and building with, and then (2) to unpacking construction in terms of seeing commonalities between the tasks and anticipating the solutions. Therefore – *iterative unpacking strategy*.

In sum, *iterative unpacking strategy* consists of a sequence of empirically-based theoretical developments so that at each step of theorizing one theory serves as an overarching conceptual framework, in which another theory, either existing or emerging, is embedded in order to elaborate on the chosen elements(s) of the overarching theory. Note that different ways to embed an additional theory into an overarching theory may exist. For example, the way of unpacking "construction" offered by Kidron, Bikner-Ahsbahs and Dreyfus (2010) is very different from the way offered by Simon et al. (2010). It is also of note that, though an overarching theory and an embedded theory are, in a way, complementary, iterative unpacking does not necessarily imply that all the constructs of the overarching theory are to be preserved. In other words, the use of iterative unpacking strategy may influence also an overarching theoretical framework, as follows: unpacking a particular aspect of a theory may shed light on the role of the rest of its aspects. For instance, the role of perturbation – one of the main concepts of the highest-level overarching theory in the above example – was reconsidered in the second iteration.

ADDITIONAL EXAMPLES

In this section an iterative unpacking strategy is illustrated by reflective accounts of two sequences of studies, in which I am involved during the last years. The first example concerns problem solving, and the second one – problem posing.

Example 1: iterative unpacking of problem solving

Problem solving as a research field is being attracting keen attention of the mathematics education community for more than 50 years. Foreground models of problem-solving are originated in seminal work of Polya (1945/1973) and developed in the eighties (e.g., Schoenfeld, 1985). Generally speaking, the models attempt to elaborate on how learners solve mathematical problems in terms of phases and attributes. For instance, a model offered by Carlson and Bloom (2005) postulates four problem-solving phases: orientation, planning, executing and checking, and operates with five problem-solving attributes: conceptual knowledge, heuristic knowledge, metacognition, control and affect.

First iteration

A heuristic knowledge component was chosen to be unpacked in the study reported in Koichu, Berman and Moore (2007). The study included a five-month teaching experiment in two Israeli 8th grade classes. The aim of the experiment was to test a particular approach to developing *heuristic literacy* in students. By *heuristic literacy* we meant an individual's capacity to use the shared names of heuristic strategies in classroom discourse and to approach (not necessarily to solve!) mathematical problems by using a variety of heuristics. Changes in students' heuristic literacy were explored in three rounds of thinking-aloud interviews conducted at the beginning, in the middle and at the end of the experiment. The interviews were based on so-called *seemingly familiar problems*, that is, problems that looked similar to the problems previously offered in the students' mathematics classrooms, whose solutions, however, were essentially different. The following problem is an example:

> The sum of the digits of a two-digit number is 14. If you add 46 to this number the product of digits of the new number will be 6. Find the two-digit number.

Indeed, at first glance the problem has a solution by means of a system of equations, as many similarly looking problems approached by the students in the classroom have had. However, composing a system of equations appears ineffective at a second glance. Such problems were used as opportunities to elicit as many heuristics as possible from the students' thinking-aloud speech without discouraging the students from the beginning by facing unfamiliar problems. The interview protocols were segmented into content units and coded in terms of 10 pre-defined heuristic processes:

> (1) Planning, including (1a) Thinking forward, (1b) Thinking from the end to the beginning and (1c) Arguing by contradiction. (2) Self-evaluation, including (2a) Local self-evaluation and (2b) Thinking backward. (3) Activating a previous experience, including (3a) Recalling related problems and (3b) Recalling related theorems. (4) Selecting problem representation, including (4a) Denoting and labelling and (4b) Drawing a picture. (5) Exploring particular cases, including (5a) Examining extreme or boundary values and (5b) Partial induction. (6) Introducing an auxiliary element. (7) Exploring a particular datum. (8) Finding what is easy to find. (9) Exploration of symmetry. (10) Generalization.

Success or failure in solving the interview problem was obviously dependant on circumstances under which the problems were dealt with as well as the whole bunch of problem-solving attributes. Consequently, the rates of success were considered irrelevant to unpacking a heuristic component of problem solving. Instead, we introduced a notion of relative *heuristic richness* of solutions. We used the following comparison criterion: One solution was called *heuristically richer* than another if the number of different heuristic processes indicated in the first solution was greater by three or more than in the second solution. Given that 10 different heuristics were considered in our study, we considered the criterion "…three or more" very demanding, and, in turn, sufficiently robust. This criterion was applied to each student individually, for comparison of her or his solutions by pairs of corresponding problems given in the first, second and third interviews. We then developed an integrative measure of individual heuristic literacy development based on the number of the pairs of the corresponding problems, in which a solution to the second problem was heuristically richer than a solution to the first one.

The measure helped us to adequately account for some of the learning effects of the teaching experiment. One of the central findings was that positive changes in heuristic literacy occurred in most of the students, yet they were unequally distributed among the students, who were defined as "stronger" and "weaker" with respect to their achievements in SAT-M (Scholastic Aptitude Test - Mathematics) administered at the beginning of the experiment. In particular, those students, who were "weaker" at the beginning of the experiment, demonstrated more significant heuristic literacy development than their "stronger" peers. We explained this result by suggesting that the heuristic content of the teaching experiment was more novel and useful for the "weaker" students, whereas the "stronger" students might have possessed the strategies prior to the experiment. The novelty of this result was in the exposure of the role and the learnability of heuristic component of mathematical problem solving, which was identified in (relative) isolation from the rest of problem-solving components. In addition, this result enabled us to formulate some pedagogical implications.

Towards the second iteration

To recap, the study quoted above attempted to unpack the heuristic component of problem solving in terms of selected heuristic processes. Heuristic literacy was chosen as an object of learning. However, though the developed measure of heuristic literacy worked well for describing some of the learning effects of the experiment, it was too simplistic in order to capture how particular heuristics are called into play. This was particularly evident when we looked at the students' solutions containing comparable numbers of heuristics, which however differed in some other respects, such as the appearance of repetitions and cycles in the students' reasoning and the nature of their decisions when facing dead ends. Thus, we feel the need for further unpacking. Specifically, we are interested in unpacking a "heuristic richness" notion. To achieve this goal, we now experiment with three interrelated ideas.

Ovadia (Ph.D. in progress) studies how particular heuristics come into play depending on how the students perceive similarities and differences between problems that were discussed in a classroom and new ones. In particular, her study focuses on the ways by which the students learn to make connections between known and new mathematical problems at the level of common heuristics needed for solving these problems. To this end her study can also be seen as an attempt to unpack the process of seeing commonalities between the tasks pointed out by Simon et al. (2010) as one of the process underlying the process of construction.

Another study (Koichu, 2010) was conducted in order to better understand why those students who possess all necessary strategic and conceptual knowledge for solving given problems sometimes miss within-the-reach solutions. In this work, I consider alternative explanations of this well-documented phenomenon in terms of three theories developed within mathematics education research, point out the limitations of these explanations with respect to a particular data set and offer an explanation in terms of the Principle of Intellectual Parsimony. The Principle states that when solving a problem, one intends not to make more intellectual effort than the minimum needed. In other words, one makes more effort only when forced to do so by the evidence that the problem cannot be solved with less effort. The explanation is built on that *efforts* in problem solving can be of a different nature.

Finally, my interest in heuristic component of mathematical problem solving and Leron's interest in proving and applying cognitive science theories to mathematics education have fruitfully intersected in our common work "Proving as problem solving: the role of cognitive decoupling" (Koichu & Leron, in prep.). In this work, we re-analyse two thinking-aloud protocols from Koichu et al. (2007) study in terms of a conceptual framework representing the increasing importance attached to working memory capacity by researchers in cognitive psychology working on problem solving and decision making. The key notion of the framework is that of cognitive decoupling, i.e., human ability to form more than one mental model of the problem situation and attempt to hold them at the same time in working memory, all the time resisting the tendency for the models to be mixed and confused. Unpacking the heuristic component of problem solving in terms of cognitive decoupling seems us instrumental for better understanding the appearance of cycles in repeated problem-solving attempts and for the use of multiple representations.

Example 2: iterative unpacking of problem posing

Koichu and Kontorovich (2012) conducted a study, in which a group of pre-service mathematics teachers was asked to pose interesting mathematical problems based on a particularly rich problem-posing situation, the Billiard Task. Our goal was to identify those traits of problem-posing processes, which are involved in the posers' attempts to formulate *interesting* problems.

A coherent conceptual framework which would be sensitive to the subtleness of the problem-posing processes and simultaneously applicable to a broad range of problem-posing tasks is not yet established. However, the literature offers several conceptualizations of problem posing, which could be utilized in our study. We decided to adopt in our study a definition of problem posing by Stoyanova and Ellerton (1996) as an overarching conceptual framework. The definition states that problem posing is "the process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems" (p. 518). This definition required a great deal of unpacking, and, as will be evident shortly, we saw this fact as an opportunity rather than as a limitation. I present here our attempts to unpack two key components of the definition: the process of constructing personal interpretations of a given situation and the notion of a mathematically meaningful problem.

Iterative unpacking of the process of problem posing

Problem posing is a natural companion of problem solving, so we decided to unpack it in terms of attributes and stages, as it had been fruitfully done regarding problem solving. In line with our earlier research (Kontorovich & Koichu, 2009; Kontorovich et al., 2012), we focused on *mathematical knowledge base*, *problem-posing strategies* and *individual considerations of aptness* as problem-posing attributes.

Mathematical knowledge base for posing problems includes the knowledge of mathematical definitions, facts, routine problem-solving procedures and relevant competencies of mathematical discourse and writing. In addition, it requires knowledge of mathematical problems that can serve as prototypes. Recalling and using a system of prototypical problems relies, in turn, on three components of the problem-solving ability, which were pointed out by English (1998): the ability to recognize the underlying structure of a problem and to detect corresponding structures in related problems, the ability to perceive mathematical situations in different ways and the ability to favour some problems over others in routine and non-routine situations. Notably, this conceptualization of mathematical knowledge base for problem posing can be seen as a result of iterative unpacking by itself.

Problem-posing strategies that emerged in our data (but had also been pointed out in prior studies) included constraint manipulation, symmetry, chaining, data-driven reasoning and hypothesis-driven reasoning. A part of our analysis was directed to unpacking the chaining strategy. *Considerations of aptness* are conceptualized as the poser's comprehensions of explicit and implicit requirements of a situation within which a problem is to be posed; they also reflect her or his assumptions about the relative importance of these requirements. Three types of considerations of aptness showed up in our data: aptness to the posers, aptness to the potential evaluators and aptness to the potential solvers of a posed problem. In the framework of Koichu and Kontorovich's study, considerations of aptness were indicated but not further unpacked. Their unpacking is one of the goals of a Ph.D. research of Kontorovich. Furthermore, one of the findings of the Koichu and Kontorovich's study was an identification and characterization of four problem-posing stages: warming-up, searching for an interesting mathematical phenomenon, hiding the problem-posing process in the problem formulation, and reviewing. Further refining and unpacking of these stages is another goal of a Ph.D. research of Kontorovich.

Interpreting a "mathematically meaningful problem" notion (a non-example)

For some time, we looked for a way of interpreting a "mathematically meaningful problem" notion among the existing ways of unpacking the closely related notions, such as "beautiful problem" and "interesting problem". However, the resulting interpretation has not been done by an iterative unpacking strategy. Thus, the chain of theoretical considerations presented below can be seen as a non-example of iterative unpacking strategy.

On one hand, the literature on aesthetic aspects of mathematics informed us that an agreement about what constitutes a beautiful problem is elusive, but offers quite stable lists of general characteristics of such problems and their solutions, such as clarity, mathematical deepness and complexity, cleverness, novelty and surprise. Second, we accepted the Crespo and Sinclair's (2008) argument that problems' descriptors such as "meaningful" belong to the rarefied discourse of mathematicians rather than that of learners. Crespo and Sinclair (2008) suggested that the learners' normative understanding of what qualifies as a worthwhile problem may develop around the notions of "mathematically interesting" or “tasty.” Third, we decided to build on the Goldin's (2002) idea that general characteristics of problems, such as "meaningful," "beautiful" or "interesting," should be seen as instantiations of one’s internal multiply-encoded cognitive/affective configurations, to which the holder attributes some kind of truth value, and not as "objective" qualifiers of the problems.

Consequently, we decided to treat in the study the descriptor "meaningful" in the manner that have been developed in past research for treating the descriptors “interesting” or “beautiful”. Namely, we operationally considered a posed problem mathematically meaningful (or interesting or beautiful) if it was evaluated as such by the poser of the problem, its readers or solvers. This decision suited our research needs, but could not be seen as unpacking, in the meaning specified in the rest of the examples. We rather bypassed delving in the cognitive and affective mechanisms underlying one's use of the descriptors and just explicated how the posed problems were operationally qualified in our data.

AFFORDANCES AND LIMITATIONS

In terms of Kuhn (1962/2012), the growing interest of the mathematics education community in networking theories might suggest that mathematics education as a research field is in transition from *pre-paradigm phase* to *normal science phase*. An iterative unpacking strategy discussed in this article is reminiscent of the accumulation-by-development strategy considered by Kuhn as the main developmental force of science during normal science periods. Indeed, the "further elaborating on…" discourse is typical for the periods of normal science, but not for the periods of paradigmatic shifts and scientific revolutions.

The presented strategy can also be seen as a particular case of the strategies of coordinating and combining (Prediger et al., 2008), the case that emphasizes accumulation of knowledge on local phenomena by establishing a specific connections between background and foreground theories. The specificity of the strategy is, in particular, in the dynamic relationship between the theories: one theory may serve as an overarching framework in one case, and as a source of conceptual tools for elaborating on elements of another theory in another case. These observations provide a background for pointing out some of the affordances and limitations of the strategy.

From a practicing researcher perspective, an iterative unpacking strategy can be instrumental for:

- situating a study in the literature and highlighting its theoretical contribution;
- wording research questions in terms of a particular conceptual framework without suppressing the possibility to further use additional conceptual frameworks in a coherent manner;

- justifying the chosen level of granularity in data analysis.

More generally speaking – here I follow the argument presented in Prediger et al. (2008) – an iterative unpacking strategy can be helpful for "better collective capitalization of research results, [adding] more coherence at the global level of the field,…, gaining a more applicable network of theories to improve teaching and learning and finally guiding design research" (p. 170).

As mentioned, one limitation of the iterative unpacking strategy is that it stops being important outside of the normal science periods (cf. Kuhn, 1962/2012). At the pre-paradigm periods, the strategies of ignoring, comparison or contrasting are typically in use. At revolutionary science periods, further elaborating on the elements of previously developed theories falls out of the mainstream. The use of the iterative unpacking strategy is limited also within the normal science periods. Briefly, all the conditions for the use of the strategies of coordinating and combining discussed in Prediger et al. (2008) apply.

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Endnote 1: The notions "theory" and "conceptual framework" are used interchangeably in this article.