NETWORKING THEORIES IN A DESIGN STUDY ON THE DEVELOPMENT OF ALGEBRAIC STRUCTURE SENSE

Thomas Janßen, Angelika Bikner-Ahsbahs

University of Bremen (Germany)

In this paper, we discuss how two different theoretical approaches are networked using a coordinating methodology with the aim to describe the development of algebraic structure sense in classroom interaction. From cycles of gathering, connecting, and structure-seeing in the social situation individual structure sense emerges in moments of objectification and subjectification, and can become visible in later instances of structure-seeing.

INTRODUCTION

The networking of theories has shown to be a helpful research frame for linking theories respecting their diversity as richness (Bikner-Ahsbahs & Prediger, 2010; Gellert, Barbé, & Espinoza, 2012). On the one hand it is a way of improving research by benefitting from the strengths of different theoretical approaches, on the other hand it may deliver an epistemological contribution to mathematics education through empirical studies on the networking of theories and through theoretical reflections on the matter and its meta-theoretical and methodological ideas (Arzarello, Bikner-Ahsbahs, & Sabena, 2009; Bikner-Ahsbahs et al., 2010; Kidron, Bikner-Ahsbahs, Cramer, Dreyfus, & Gilboa, 2010). Radford (2008) postulates the semiosphere as a space for the networking of theories, where theories are distinguished by their identities and boundaries towards other theories. Theories can then be seen "as a way of producing understandings and ways of action based on the triple (P, M, Q)" (Radford, 2008, p. 320) where P is the set of basic principles, M in this pragmatic approach is a set of methodologies as a way of data collection and interpretation based on P and related to Q, while Q is a set of paradigmatic questions deeply related to P and M. However, methodology in our view also includes the meta-rules connected to P that imply specific ways of data collection, analysis, and interpretation. Concerning this definition, networking takes place between these three components. The degree of integration through networking may vary according to the *networking strategies* from understanding other theories and making understandable the own theory to coordinating or integrating theories locally and synthesizing them (Bikner-Ahsbahs & Prediger, 2010, 495 ff.). In this paper an example of empirical research exploring the strategy of coordinating is presented. It is part of a design study on the development of algebraic structure sense in grade 8. The specific contribution of two coordinated theories is pointed out focusing on the boundary between the theories and on the process of crossing it during the research that leads to a deepened insight into the development of structure sense.

RESEARCH BACKGROUND: ALGEBRAIC STRUCTURE SENSE

The research literature on algebra education has stated a demand for a better support of students' *algebraic structure sense* in many instances—most explicitly Linchevski and Livneh (1999), similarly Arcavi (2005). However, the only systematic attempt thus far to define such a sense has been undertaken by Hoch (2007, also see Hoch and Dreyfus, 2004). In her operational definition of algebraic structure sense for high school algebra (Hoch, 2007, p. 40) she refers to algebraic structures as a defined set of terms and expressions, which can be worked on using appropriate operations. On this basis, specific forms of algebraic structure sense are ascribed to a student, if he or she can recognize a familiar structure in its simplest form, deal with a compound term as a single entity and through an appropriate substitution recognize a familiar structure in a more complex form, and, most sophisticated, choose appropriate manipulations to make best use of a structure. Based on this definition, Hoch in a first step shows that many students lack the described competencies and then demonstrates that through individual training a sustaining improvement can be accomplished. Hypotheses are presented regarding the factors that have an effect on the individual's acquisition of algebraic structure sense.

For everyday classroom instruction Hoch proposes to let the students work on tasks only vaguely described as "adapted to make them suitable for group work", the teacher could then help if necessary (Hoch, 2007, p. 133). However, both the construction and the implementation of the learning environments may be a very difficult task for many teachers, considered that almost half the students don't show structure sense in the tests, and that many teachers neither do (Hoch & Dreyfus, 2004, p. 55). Deeper insight into the development of algebraic structure sense is needed. The question thus is how students develop algebraic structure sense in everyday classroom interaction. The shift in attention from states to processes implies a shift in the theoretical understanding of algebraic structure sense: With a static definition referring to a hypothetical final product of algebra instruction no reliable insights will be gained about the processes through which algebraic structure sense develops. This leads to the approach of investigating the development of algebraic structure sense by means of networking two appropriate theories.

THEORIES CONSIDERED

To investigate structure sense as an ever developing attitude rather than as a spotty competence was inspired by the GCSt epistemic action model that is part of Bikner-Ahsbahs' (2005) theory of interest-dense situations. It assumes seeing structures as the last of three collective epistemic actions performed to answer a mathematical question. Preceded by the *gathering* of smaller units of knowledge and their successive *connecting*, *structure-seeing* means the perception of mathematical structures, represented by regularities, rules, or exemplary solutions (Bikner-Ahsbahs, 2005, p. 202, our translation). While seeing structures, students refer to a unity of relations built on gathered and connected mathematical meanings. Theoretically, students may see new structures or known structures within new mathematical task contexts when producing mathematical meanings through gathering and connecting is saturated. This definition of structures allows for an open-minded view on algebraic structures as they are learned in school, defined rather by the possibilities of the learners than by the curriculum. The background theory for this approach is social constructivism linked with an interpretative view according to which people construct meanings about things through interpreting them in the situation. Thus, the epistemic actions are reconstructed through the interpretation of the students' utterances.

A basis for a deeper understanding of the connection between collective structure-seeing and the development of individual, enduring structure sense can be found in the *activity theory* as proposed by Leont'ev (1978). According to activity theory, the individual's *personality* develops in *activity* under the given cultural conditions. Activity in this context does not mean some undirected motion but is characterized by its inherent *motive,* e.g. to solve an equation. On lower levels one can find *actions* directed to more specific *goals* and *operations* which are determined by the *conditions* under which they are performed. This set of ideas has been made fruitful for mathematics education by Roth and Radford (2011; regarding algebraic structures in particular see Radford, 2010). Their dialectic account describes mathematical meaning-making as *objectification*, i.e. the disclosure of an activity motive to the learner, and *subjectification*, i.e. the development of personality as reordering of the subject's structure of motives in the very same process. Regarding methodology, an activity-theoretical account calls for a deep analysis of the societal, social, and individual processes. Roth and Radford concretize this by arguing for a semiotic approach.

Regarding the research question, the GCSt model may offer a clarification of the situational and social prerequisites of objectification/subjectification, where structure sense develops as a stable feature of the individual student's personality. A more detailed empirically based description of this interplay of the theories is the subject of this article. This first requires a consideration of methodology and method.

METHDOLOGY AND METHOD OF THE EMPIRICAL STUDY

The intended development of both theory and practice is precisely the kind of setting that *design research* has been developed and proven useful for over the last two decades (for an overview see van den Akker, Gravemeijer, McKenney, & Nieveen 2006). In a broad definition, the intent of this approach "is to investigate the possibilities for educational improvement by bringing about new forms of learning in order to study them" (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003, p. 10). This design study took place between December 2011 and July 2012 in a grade 8 class of a comprehensive school in a socioeconomically deprived area of Bremen, Germany. Three factors led to this choice: First and most importantly, the teacher had to be interested in the reflection and improvement of her teaching. Second, a certain degree of heterogeneity was appreciated, for students' problems with algebraic structures are very likely to differ with school performance, however intense and directed the causality may be. The third argument was about the grade to investigate. Due to the given curriculum (Die Senatorin für Bildung und Wissenschaft, 2010) we could hope for a variety of algebraic structures that students can acquire a sense for in grade 8.

In this paper, learning processes regarding linear equations and linear functions are considered. The teacher was committed to patiently leave room for the learning process to take place, as this is regarded helpful not only in the theory of interest-dense situations, but also in Arcavi's (2005) considerations regarding symbol sense. In both cases the students were encouraged to make on-hand experiences in the initial phase of the unit. For the introduction to linear equations a variant of the matchbox algebra as presented in some German-language textbooks (first of all Affolter et al., 2003) was implemented. In this approach equations are materialized by matches (representing integers) and matchboxes (each holding the number of matches that is the solution of the equation, see fig. 1a). Already at this early point a focus was laid on the equations' structures. The idea was that there is a natural impetus to structure a pile of unsorted matches and boxes. Linear functions were introduced referring to the changing water level in a straight container steadily filled or emptied. The anchor representation of steady change with an initial water level was consciously chosen based on analyses of the experiences previously made in the unit on linear equations.

Qualitative analyses were conducted focusing on the cases where structure-seeing takes place. The goal was to clarify the connection between structure-seeing and objectification/subjectification. Assuming with Roth and Radford (2011, p. 25) that "subjects of activity … exhibit to each other whatever is required to pull off an event as that which it is", the analyses built on a comprehensive record of the classroom interaction and the expressed thinking of the students. It includes photographs as well as the written texts of the students, and, as the anchor of analysis, video footage. One camera provided an overview of the class, while two selected pairs of students were filmed by one video camera each to document their long-term learning process in depth. Each group included a boy and a girl, and a high and a low achiever according to the levels defined by the local board of education. Furthermore, the students were selected to represent a range in language proficiency as valuated by the teacher.

Networking strategies: an approach for linking two theories

The initial networking strategy was combining, i.e. juxtaposing the theories in order to understand a specific data set better from each perspective. This implies clarifying what structure-seeing means in the current epistemic process from the view of the GCSt model on the one hand, and on the other hand how the motive that constitutes objectification could be observed. Our research question thus led to the following *coordinating* analysis method that links the two theories' methodologies: a situation of structure-seeing is identified, the corresponding episode is transcribed; and then the epistemic process of this episode is analyzed by the use of the GCSt model. This way, the specific structure seen by the students is captured. Based on this, hypotheses about the motive of objectification and its relation to the development of structure sense are expressed and then validated by the successive data (methodical link).

Before performing a developmental cycle of analysis, the a priori-status of the design theory will be described. It serves as a hypothetical model that illustrates the coordination of the two theoretical approaches and will be the basis to further the design theory through data analysis.

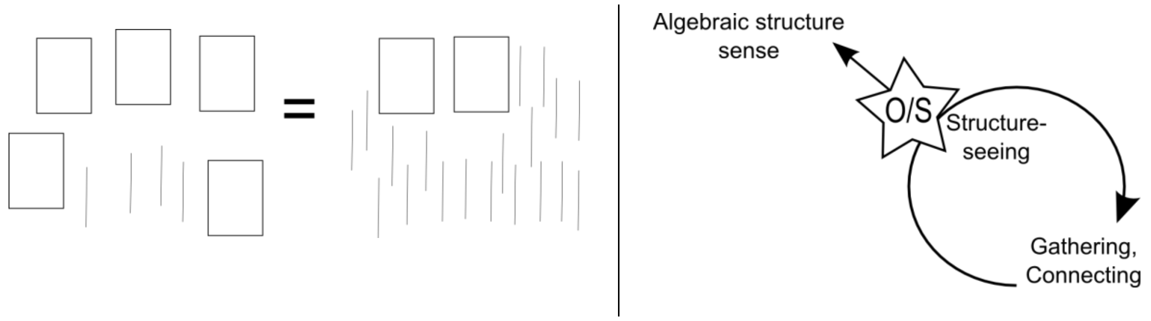


Fig. 1a (left): The matchbox equation the students worked on in the scene described below; Fig. 1b (right): The moment of objectification (O) as a special case of structure-seeing where algebraic structure sense develops through subjectification (S).

A HYPOTHETICAL MODEL AS A CASE OF COORDINATING

Coordinating theories is performed "when a conceptual framework is built by well fitting elements from different theories" (Bikner-Ahsbahs & Prediger, 2010, p. 491). This is especially used in the building of conceptual frameworks when complementary insights are approached. Hence, the networking strategy of coordinating is appropriate in the networking case presented in this paper.

In activity theory, one can assume that through processes of objectification the students change their view of the algebraic world. Subjectification, being the development of the student's personality in this very process, can thus describe the development of algebraic structure sense. Algebraic structure sense in these terms is a developing attitude towards specific algebraic structures, for example linear equations or linear functions. However, the theory of objectification remains unspecific about fostering conditions for objectification (and thus the development of algebraic structure sense). From the theoretical perspective of the GCSt model, objectification can be seen as a special case of structure-seeing. This given, cycles of gathering, connecting, and structure-seeing are the soil for the development of algebraic structure sense. These processes may go on for some time until the specific moment of objectification. This moment constitutes a substantial change in quality: the students not only see the entity, they are also able to use it driven by the disclosure of the motive. The development of personality also passes through this change in that the structure of motives is reorganized towards the inclusion of the new motive that the students become aware of.

The connection between the models at the point of objectification deserves deeper consideration. Two states concerning a specific activity can be distinguished: in performing cycles of gathering, connecting, and structure-seeing, students look at the task with a tendency to perceive all kinds of structures in the situation. These are not necessarily mathematical and correct. After the point of objectification the students' attitudes have changed, they are able to use the specific structure within the activity and work it out. Before objectification the GCSt model is the foreground model to investigate epistemic processes that can lead to objectification. After the instance of objectification the GCSt model has taken the back seat and the dialectic of objectification and subjectification comes to the fore providing the frame to reconstruct the development of algebraic structure sense as a general attitude of the subject. With this model (fig. 1b) in mind the development of structure sense before and after the instance of objectification can now be investigated through a coordinating analysis.

COORDINATING ANALYSIS AT THE MOMENT OF OBJECTIFICATION

The first scene presented here shows how the model presented above helps understand the crucial moment of objectification. In this case, it is supported by a metaphor that the teacher provides. Previously, the students have tried to keep "it equal, but more simple". However, the "it" is not clear. As part of the riddle only the mutual equality of the two sides must be provided, but the students additionally try to keep each side materially equal over time. In a first step they relax this rule by exchanging matches and boxes between the two sides, but notice that this does not bring any improvement of the situation. This material equality concept keeps the students stuck, and first the teacher is so with them, but then she gives a decisive hint:

1 /Teacher: okay so what can one do now on both sides (*taps the table on both sides*) so that it gets clearer you always have to do the same. (*H draws some dirt off the table, T looks at S, S looks at the table, still having three boxes in her hand*) (…) (*gets up and walks away, whispering in Sabine's ear*) ,take away something.

Transcription key:

w-e-l-l	speaking slowly
exact.	dropping the voice
exact'	raising the voice
exact-	voice kept in suspense
,exact	with a new onset
EXACT	with a loud voice
(.),(..),...	1, 2, ... sec pause
(7sec)	7 sec pause
/Karl:	interrupts the previous speaker
SX:	unidentified student

2 Sabine: (*looks at H, leaves one of the three boxes on the table and takes the two closer to herself*) Take away something

3 Herbert: A-h- ,man are you smart. (*S laughs, puts the boxes on the table again, slightly left of the others*) ,Mrs. Kahn is totally mean. ,take away something okay-

4 Sabine: (*looks at H, points at the right side*) so if we nine are five ,ah just take away four' (*both students each take away four matches from the right side, H to the bottom right, S towards the equal sign*)

5 Herbert: One two three four I have.

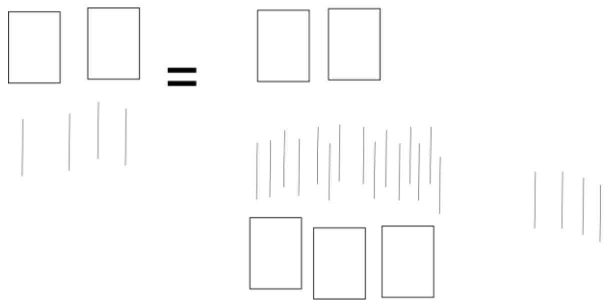
6 /Sabine: I have taken away here-

7 /Herbert: a-h-r. (*puts his matches back, S laughs*) ,are fifteen.

8 Sabine: Yes. ,and fifteen- (*points at the right side, looks at the left side*) (...) (*points at the five matchboxes on the left side one by one*) ,one two three four five- (*takes three of the boxes and puts them further to the middle*) ,divided by three- ,are five.

9 Herbert: Yes. (*looks at S, smiles*)

10 Sabine: And now we have (*puts the three boxes beneath the matches (see figure on the right), first points at the matches and then at everything around the equal sign*) this here and the three (*moves backwards with her chair*) we (*claps her hands*) HAVE IT. (*H looks at the table in front of him, S leans forward again*) ,look. ,because now you (plural) have (*first points at the 15 matches, then at the three boxes*) ,if these were in here (*looks at H*) ,right' (*points at the four matches put aside*) we still have four outside.



11 Herbert: Eheh'

12 Sabine: (*first points at the 15 matches, then at the three boxes*) And those are in there right. (*H looks at S, first grinning broadly, then with his mouth closed, then grinning again*) (4sec) ,do you think that works' (*puts the three boxes on the left side again*) (...)

13 Herbert: Well once more. (*puts his right hand on the right side*) ,there are fifteen.

14 /Sabine: (*takes the three boxes on the left side in her hand and shakes them while talking*) yes and her are also fifteen then because fifteen divided by three are five

15 /Herbert: (*joins in, holds up his right hand with all fingers up*) divided by three are five.

Sabine says “Yes” and calls the teacher (who is busy with other students). The students look at each other. “whoo” they shout, raise their arms and Sabine says: “Herbert we are good.” Then the two students turn to a neighboring table and Sabine starts explaining her solution.

The metaphor „take away“ (1-4) enables the students to overcome the material equality concept and to rivet on solving the equation: having permission to “take away“ they can ignore the parts materially identical on both sides (four matches and two boxes) (8-10). The remaining three boxes and 15 matches are connected by bringing them spatially together (10). The students see the divisibility by three and repeatedly check their result (10-14). Hence, the possibility of taking away helps seeing structures and experiencing objectification. The emotionality of the scene hints at both the leap the students do regarding the structure of linear equations and at the change in their personality regarding this specific situation.

In the second scene presented here we can now see how the motive from the objectification described above is *conserved* in *key situations* expressed by the metonym "matches". It takes place five months after the first scene and more than three months after the last time the class worked on equations, when the (higher track) students are asked to identify the moment in which the water level in two bathtubs is the same, when one of them starts at a higher level but drains at a higher speed. In the group the students and the teacher have argued that to find out the unknown, for which the terms of the functions deliver the same value, the terms have to be equal. This results in an equation on the blackboard, which is not named as such. Now the teacher asks:

/Teacher:	okay so what is that here now. (*briefly points at the blackboard, Rico puts his hand up, T briefly points at him*)
Rico:	(incomprehensible)
Teacher:	That are two terms yeah' (*many students talk, incomprehensibly*)
Sabine:	(*wags her right index in the air*) ISN'T THAT THE THING WITH THE cigarette things ,matches- (*T as well as Herbert and Ahmed look at her, the rest of the group looks at the blackboard*)
/Teacher:	p-s-t- ,o-h-
SX1:	equation'
Teacher:	(*looks at SX1, still points at the equation with her right hand*) An equation. exactly.
SX2:	A-h-

This is a case of structure-seeing in a new context. After a long period where equations did not play a role in the classroom, the word "matches" activates all the procedures that were developed based on the matchbox environment. Especially the repeated use of the metaphor "take away" in the following collective solution of the equation indicates the importance of the situation in which the earlier objectification took place (cf. Lakoff & Johnson, 1996). That means that in objectification knowledge can be conserved and made long-term-accessible in key situations. Structure-seeing activates this knowledge, not only for an individual, but available for the whole group by a metonym substituting the whole situation. This coordinating analysis expands the hypothetical model by the aspect of *conserving knowledge* and makes clear how previous processes of objectification/subjectification can again become part of remote discussions in the classroom.

REFLECTIONS ABOUT THE NETWORKING PROCESS

In this contribution two theoretical approaches were networked by a coordinating methodology while investigating the impact of structure-seeing for the development of structure sense through objectification. Objectification was characterized as a form of structure-seeing, a concept at the boundary of both theories. With the GCSt model the preparation of this moment can be investigated. Activity theory according to Radford and Roth provides the tools which capture how structure-seeing as a collectively constituted action impinges the development of algebraic structure sense. Objectification seems to initiate a leap of quality: disclosing a motive it immediately enlarges the potential to act. The simultaneous emotional and valuing expressions indicate subjectification. Objectification seems to be experienced so deeply that it is conserved in the situation and can be activated again in further situations, which are also prepared by gathering and connecting. Using the described coordinating analysis, objectification can be worked out more comprehensibly in the course of further analyses. However, it will be seen how far this kind of networking through coordinating two concepts methodologically will lead towards local integration.

REFERENCES

Affolter, W., Beerli, G., Hurschler, H., Jaggi, B., Jundt, W., Krummenacher, R., Nydegger, A., Wälti, B., & Wieland, G. (2003). *mathbu.ch 8. Mathematik im 8. Schuljahr für die Sekundarstufe I*. Bern: Schulverlag blmv & Zug: Klett und Balmer.

Arcavi, A. (2005). Developing and using symbol sense in mathematics. *For the Learning of Mathematics 25* (2), 42–47.

Arzarello, F., Bikner-Ahsbahs, A., & Sabena, C. (2009). Complementary networking: enriching understanding. In V. Durand-Guerrier, S. Soury-Lavergne, & S. Lecluse (Eds.), *Proceedings of CERME 6*. Lyon, France. Retrieved August 23, 2010 from http://www.inrp.fr/publications/edition-electronique/cerme6/plenary-01-bikner.pdf, published on CD.

Bikner-Ahsbahs, A. (2005). *Mathematikinteresse zwischen Subjekt und Situation. [Interest in mathematics between Subject and Situation]*. Hildesheim, Berlin: Verlag Franzbecker

Bikner-Ahsbahs, A., Dreyfus, T., Kidron, I., Arzarello, F., Radford, L., Artigue, M., & Sabena, C. (2010). Networking of theories in mathematics education. In Pinto, M. M. F. & Kawasaki, T. F. (Eds.), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education*. Vol. 1 (pp. 145-175). Belo Horizonte, Brazil: PME.

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of theories—an approach for exploiting the diversity of theoretical approaches. In B. Sriraman &

L. English (Eds.), *Theories of mathematics education: seeking new frontiers* (pp. 479-512). New York: Springer.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher 32* (9), 9-13.

Die Senatorin für Bildung und Wissenschaft (2010). *Mathematik. Bildungsplan für die Oberschule*. Retrieved January 11, 2012 from http://www. lis.bremen.de/sixcms/media.php/13/2010_BP_O_Ma%20Erlassversion.pdf

Gellert, U., Barbé, J., & Espinoza, L. (2012). Towards a local integration of theories: codes and praxeologies in the case of computer-based instruction. *Educational Studies in Mathematics*, Online First. DOI 10.1007/s10649-012-9427-5.

Hoch, M. (2007). *Structure sense in high school algebra*. Unpublished doctoral thesis. Tel Aviv: Tel Aviv University.

Hoch, M., & Dreyfus, T. (2004). Structure sense in high school algebra: the effect of brackets. In M. Johnsen Høines, & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* Vol. 3 (pp. 49–56). Bergen (Norway): PME.

Kidron, I., Bikner-Ahsbahs, A., & Dreyfus, T. (2011). How a general epistemic need leads to a need for a new construct: A case of networking two theoretical approaches. In Pytlak, M.; Rowland, T. & Swoboda, E. (Eds.), *Proceedings of the 7th Congress of the European Society for Research in Mathematics Education* (pp. 2475-2485). Rzeszów: University of Rzeszów.

Lakoff, G., & Johnson, M. (1996). *Metaphors we live by*. Chicago: University of Chicago Press.

Leont'ev, A. N. (1978). *Activity, consciousness, and personality*. Englewood Cliffs: Prentice-Hall.

Linchevski, L., & Livneh, D. (1999). Structure sense: the relationship between algebraic and numerical contexts. *Educational Studies in Mathematics 40*, 173–196.

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *ZDM-International Journal 40*, 317-327.

Radford, L. (2010). The eye as a theoretician. Seeing structures in generalizing activities. *For the Learning of Mathematics 30* (2), 2–7.

Roth, W.-M., & Radford, L. (2011). *A cultural-historical perspective on mathematics teaching and learning*. Rotterdam: Sense.

Van den Akker, J., Gravemeijer, K., McKenney, S., & Nieveen, N. (Eds.) (2006). *Educational design research*. London, New York: Routledge.