TWO PEDAGOGICAL APPROACHES LINKING CONCEPTUAL AND PROCEDURAL KNOWLEDGE

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There are two fundamental dilemmas that appear throughout human life: (1) Do we have to understand being able to do, or vice versa? and (2) Should things be emphasized as objects or as processes? These dilemmas can be converted to the question of the balance between conceptual and procedural knowledge. This article represents two research-based and empirically tested approaches: the educational approach emphasizing conceptual knowledge and the developmental approach stressing procedural knowledge. Pedagogical implications of these approaches are discussed within tensions between different paradigms.

INTRODUCTION

ATCM 2008 plenary (Haapasalo, 2008) emphasized the following tensions when developing research-based theories for instructional praxis: (1) Objectivism vs. Radical Constructivism, (2) Developmental Approach emphasizing procedural knowledge vs. Educational Approach stressing conceptual knowledge, (3) Gagne's Systematisation emphasizing guided learning vs. Minimalist Instruction emphasizing student's volition to learn, (4) Instrumentation where technology is shaping the actions of doing mathematics vs. Instrumentalisation where technology is shaping also the mathematical objects, (5) Learning by Instructional Materials vs. Learning by Design, (6) Teaching mathematical contents vs. Emphasizing sustainable heuristics from the history of mathematics, and (7) Looking at internal problems of mathematics education vs. Applying business principles to shift the bad reputation of mathematics. These tensions can be converted into the following *Challenges*:

Ch1: Solid theories for collaborative social constructions,
Ch2: Solid theories to link conceptual and procedural knowledge,
Ch3: Dilemma between systematisation and minimalism,
Ch4: Relating instructional design and assessment to instrumental genesis,
Ch5: Learning by Design,
Ch6: Revitalizing sustainable heuristics, and
Ch7: Applying business principles to shift the bad reputation of mathematics.

This article represents responds to *Ch2* and discusses pedagogical implementations taking into account some key points of the other tensions with their challenges. To justify the characterization of the two knowledge types in this article, let us first pick up some findings of the literature analysis of Haapasalo and Kadijevich (2000).

• According to Ivic (1991), Piaget made a distinction between 'practical knowledge' and 'conceptual knowledge', whereas Vygotsky dealt with three levels of knowledge: 'manifest content', 'instrumental knowledge' and 'structural knowledge'.

• For Anderson (1983) procedural knowledge comprises condition-action rules, whilst 'declarative knowledge' is composed of tangled hierarchies of cognitive units.

• Nesher (1986) made a distinction between learning algorithms and learning for understanding, pointing out that 'algorithmic performance' and 'understanding' can only be examined separately after the learning has been completed.

• Hiebert & Wearne (1986) emphasized that procedural knowledge is rich in algorithms for completing tasks but is lacking in relationships, whereas conceptual knowledge is rich in relationships but is lacking in algorithms for completing tasks.

• Skemp (1987) proposed 'instrumental understanding' referring to the ability to utilize rules without knowing why they work, 'relational understanding' referring to the ability to infer particular rules or procedures by considering some general relationships, and 'logical understanding' denoting the ability to reason deductively.

• Gray & Tall (1993) define 'procept' as "a combined mental object consisting of a process, a concept produced by that process, and a symbol which may be used to denote either of both", and introduce 'procedural thinking' and 'proceptual thinking'.

• Sfard (1994) distinguished between 'operational thinking' and 'structural thinking'. While the former deals with processes in terms of operations on objects, the latter refers to objects made out of these processes.

In addition to this cavalcade, it is appropriate to mention that Rittle-Johnson, Siegler and Alibali (2001) define 'procedural knowledge' as the ability to execute action sequences to solve problems, whilst 'conceptual knowledge' is implicit or explicit understanding of the principles that govern a domain and of the interrelations between units of knowledge in a domain. They suggest that these two knowledge types develop interactively, and the key mechanism underlying these relations is change in problem representation. However, their terminology 'conceptual understanding' and 'procedural skill' could be questioned as Devlin (2007) does, for example. To avoid the view that procedural knowledge would be dynamic but conceptual knowledge would be static, Haapasalo and Kadijevich (2000) suggest a characterization that is used to discuss the topic of this article because it fits constructivist paradigms of teaching and learning.

DYNAMIC VIEW OF CONCEPTUAL AND PROCEDURAL KNOWLEDGE

The two knowledge types are characterized as follows:

• *Procedural knowledge* (denoted by ***P***) denotes dynamic and successful utilisation of particular rules, algorithms or procedures within relevant representation forms. This usually requires not only knowledge of the objects being utilized, but also the knowledge of format and syntax for the representational system(s) expressing them.

• *Conceptual knowledge* (denoted by ***C***) denotes knowledge of and a skilful "drive" along particular networks, the elements of which can be concepts, rules (algorithms, procedures, etc.), and even problems or problem fields (a solved problem may introduce a new concept or rule) given in various representation forms.

The dynamic view of ***C*** fits modern theory of neural networks allowing sophisticated and complicated problem solving through "skilful drive". This means refusing the view of Rittle-Johnson, Siegler and Alibali (2001) that ***P*** would be the only knowledge type when executing action sequences to solve problems, and ***C*** would mean "understanding of the interrelations between units of knowledge in a domain". However, the author agrees with their view that the mechanism underlying the relation of ***P*** and ***C*** is change in problem representation.

The characterization above does not contradict the view of Star (2007) that both knowledge types can appear for a person either on a superficial or a deep level. ***P*** often calls for automated and unconscious steps, whereas ***C*** typically requires conscious thinking. However, the former may also be demonstrated in a reflective mode of thinking when, for example, the student skilfully combines rules without knowing why they work. Like the concept of 'problem', the ***P*** vs. ***C*** knowledge distinction is at least person, content and context dependent. As regards educational context, it depends on the pedagogical theory guiding the learning process (cf. *Challenges* above). This situation is immediately realized when suitable tasks are looked for. These aspects are discussed in Haapasalo and Kadijevich (2000).

TWO PEDAGOGICAL APPROACHES

Based on the logical relation between ***P*** and ***C***, two pedagogical approaches are defined: *developmental* and *educational*. The first one is based upon the *genetic view* (i.e. ***P*** is necessary but not sufficient for ***C***) or the *simultaneous activation view* (i.e. ***P*** is necessary and sufficient for ***C***). The logical basis of the second one is *dynamic interaction view (*i.e. ***C*** is necessary but not sufficient for ***P***), or the simultaneous activation view again. The latter means that the learner has opportunities to activate simultaneously conceptual and procedural features of the object.

A literature analysis reveals the dominance of ***P*** over ***C*** in the development of scientific and individual knowledge (cf. Haapasalo & Kadijevich, 2000). Regarding the phylogenesis of mathematical knowledge, ***P*** has developed faster than ***C*** because early mathematicians were pragmatically oriented. Ivic (1991) underlines that for Vygotsky, scientific knowledge is based upon procedural and conceptual components, and for Piaget, ***C*** comes developmentally later and is based on practical ***P***. Furthermore, for both Piaget and Vygotsky, an awareness of one's own thinking processes is related to "mature stages of development and to the appearance of conceptual systems of knowledge." This analysis clearly emphasizes that it is the presence of metacognition that is crucial to ***C*** development and that for this reason acquisition of ***P*** is generally more accessible.

We know from the basics from cognitive psychology that our world is a world of meanings, not a world of stimuli. If adapting constructivist paradigm of teaching and learning, instead of 'learning environments' we should speak about 'investigation spaces' appearing psychologically meaningful for students. An interactive manipulation at computer screen, for example, can be more “real” than what is conventionally called “real world”. This implies the need to apply a developmental approach in the instructional design: students should have opportunities to go for their more or less spontaneous ***P***. On the other hand, a very crucial educational goal in a modern society is to scaffold citizens' abilities to identify and construct links within complicated multi-causal and multi-disciplined knowledge networks. This means investing on ***C***, even in such a way, that students also learn appropriate procedural skills. Thus, the educational approach seems to cause a conflict with developmental approach. However, based on large empirical data and sophisticated statistical analysis, the recent dissertation of Lauritzen (2012) reveals that actually both approaches should be combined.

LAURITZEN STUDY

In many cases difficulties for students are caused from the fact that mathematical things should be understood simultaneously both as process as concepts, being so-‘procepts’ in the sense of Gray & Tall (1993). Function is one of the best examples of procept and hence an especially interesting research object. There are few studies of how these two knowledge types of function relate to each other and what could be an appropriate pedagogical implication. Even to find an instrument to measure ***P*** and ***C*** independently from each other appears to be a hard task (cf. Haapasalo & Kadijevich 2000 and Kadijevich & Haapasalo 2001).

Lauritzen explored how ***P*** and ***C*** of functions can be measured, what is the relationship between them, and how the students' ability to apply functions within economic and other mathematical tasks depends on the two types of knowledge. The outcome was related to the pedagogical philosophy applied to the study population at the upper secondary school. Data was collected at three different stages from 476 students in economics. Confirmatory factor analysis was applied to develop tasks to measure three components: ''procedural knowledge of functions', 'conceptual knowledge of functions' and 'the ability to apply functions'.

The study revealed that a large group of subjects scored well in ***P*** but modestly in ***C***. Scores in ***C*** appeared even lower among those subjects who showed poor ***P***. However, all students who scored high in ***C***, scored also high in ***P***. Thus, the results support the genetic view. On the other hand, ***P*** alone seems to be insufficient for the student to be able to apply functions. The educational background of the subjects might have fostered this outcome. Interviews indicated that the focus of the school teaching has been on simple procedures without links to abstract ***C***.

SOPHISTICATED INTERPLAY BETWEEN TWO APPROACHES

Now that the fundamental theory of author's *MODEM project* (*M*odel *C*onstruction of *D*idactical and *E*mpirical Problems of *M*athematic Education; see Haapasalo, 1993)[1] has found a rigorous reassurance by Lauritzen, it is appropriate to represent it in more detail by considering the conceptual field ***C*** [Proportionality ($\boldsymbol{C}_{\mathrm{prp}}$), gradient of a straight line through origin ($\boldsymbol{C}_{\mathrm{grd}}$), "Depends Linearly on" ($\boldsymbol{C}_{\mathrm{lin}}$)]. The abbreviations are used to stress that those three concepts differ actually just by verbal (*V*) descriptions, as the symbolic (*S*) and graphic (*G*) expressions are exactly the same ones. Figure 1 illustrates an appropriate pedagogical meta-strategy when trying to scaffold students in constructing conceptual knowledge ***C***. Dynamical links between representation forms of gradient (V_{grd}, G_{grd}, S_{grd}) are constructed, at first. After that, the student can easily link two new verbal expressions V_{grd} and V_{lin} to his or her dynamical cognitive network by just renaming them. So, the umbrella approach is definitely educational. However, to reach the conceptual field ***C*** in Figure 1, we have to find a starting point that is psychologically meaningful for the student and which can be understood by using spontaneous procedural knowledge (i.e. applying developmental approach). For this purpose, the everyday concept of 'slope' can be used instead of 'gradient', at first. Figure 2 illustrates interplay between the two approaches within the MODEM -framework.

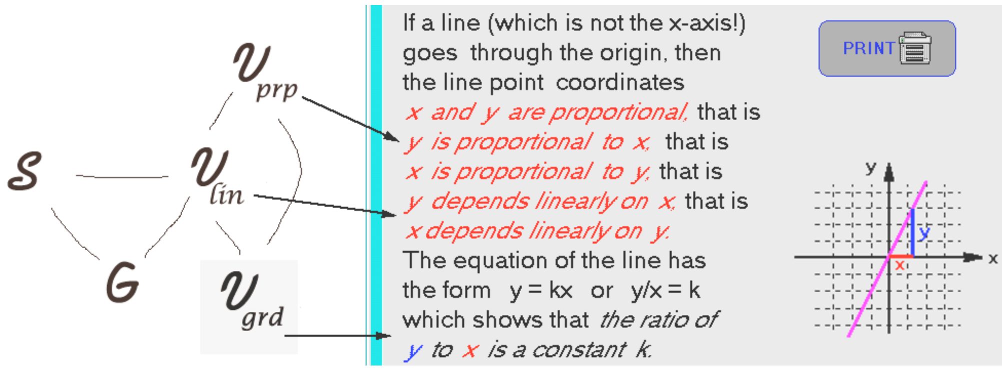

Figure 1. Extending the conceptual knowledge $\boldsymbol{C}_{\mathrm{grd}}$ to *C* within educational approach.

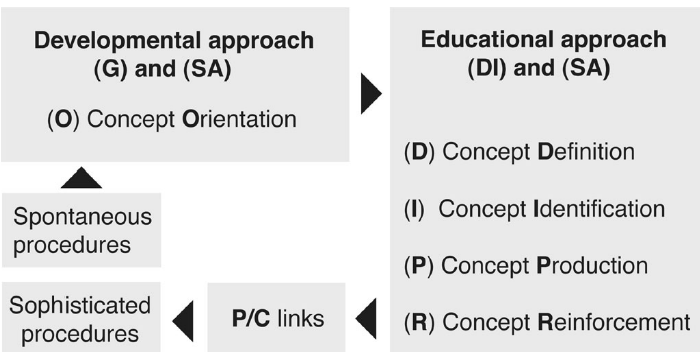

Figure 2. Interplay between developmental and educational approach.

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[1] See http://wanda.uef.fi/lenni/modemeng.html, and regarding Figures 1 and 2: http://wanda.uef.fi/lenni/programs.html

When planning a constructivist approach to a certain concept, the focus is on the left-hand side of Figure 2. On the other hand, when offering students opportunities to construct links between representation forms, the focus is on the right-hand box, which describes the stages of mathematical concept building.

*Concept Orientation (**O**)* forms the first phase of the quasi-systematic concept building. It basically utilizes a developmental approach: the interpretations of the situation can be based on mental models of the pupils, coming, more or less, from their naïve procedural ideas. These act like a wake-up voltage in an electric circuit that triggers another, much more powerful current to be amplified again. The procedural and conceptual knowledge types start to support each other, offering a nice opportunity to use the principle of simultaneous activation, for example. This principle, being at the intersection of the logical definitions of the two approaches, links the developmental approach and educational approach in a most natural way.

The role of the *Concept Definition (**D**)* is to offer students opportunities to make their own investigations, to express results especially in verbal forms in each case, and to argue about these results within the collaborative teams and between the teams. As a result of social construction, a definition for the concept is born, meaning that students try to fix the relevant determiners of the concept in verbal, symbolic and graphic forms. Thus, we notice how the response to *Ch1* (cf. Figure 1) is embedded within the knowledge construction, including interesting pedagogical variables. The next phases of concept building utilize the principle of dynamic interaction.

In the phase of *Identification* (***I***) we have to give students opportunities to train them selves in identifying concept attributes in verbal (*V*), symbolic (*S*) and graphic (*G*) forms. For this we need six kinds of tasks (I): ***IVV***, ***IVG***, ***I**V*S, ***IGG***, ***ISS*** and ***ISG***. During the learning process, the teacher must be ready, if necessary, to begin with tasks that require distinguishing between only two elements before going on to the identification of several elements.

In the phase of *Production* (***P***) we have to give pupils the possibility to produce from a given presentation of the concept another representation in a different form. The development of production (P) requires nine combinations: ***PGV***, ***PGS***, ***PGG***, ***PSG***, ***PSV***, ***PSS***, ***PVS***, ***PVV*** and ***PVG***. The tasks of identification and production must be achievable without any complicated processing of information on the student's part.

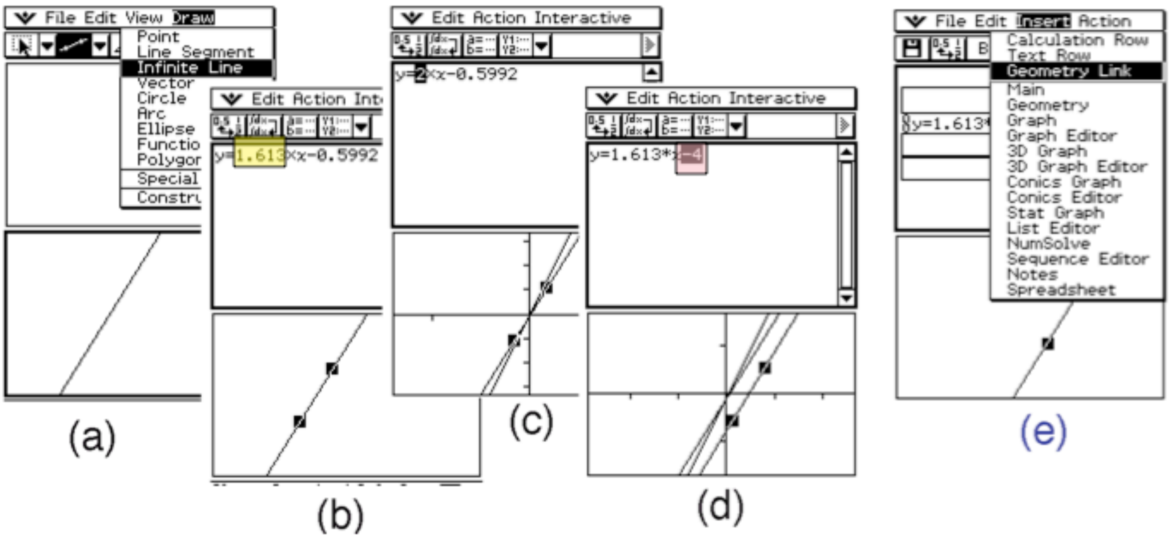
In the phase of *Reinforcement* (***R***), the goal is to reinforce and utilize the concept attributes, to develop procedural knowledge to be used in problem solving and applications, and extend the concept to a conceptual field. After constructing appropriate links between representation forms *S, G* and V_{grd} (i.e. using the term 'gradient' in the verbalisations), this knowledge structure can be easily extended to the conceptual field ***C*** in Figure 1 by linking two other verbalisations V_{prp} and V_{lin} (i.e. the terms 'proportional' and 'depends linearly'). In computer terminology this would mean nothing else than a "rename command".

The MODEM- framework turned out to be suitable to design conceptual tasks for assessment as well learning. Several experimental studies revealed, among others, that (1) for most students, concept building was a long process in which the five phases (of the previous page) could be utilized systematically and successfully, (2) the identification phase played a central role in concept building, and seemed to create a pleasant learning environment for the formation of concept attributes, (3) ***C*** was most reliably measured by the production tasks, which were usually difficult for most students, especially the ***PVV*** type, (4) the framework probably promoted also ***P-C** links,* because a significant positive correlation was found between students' scores on those task types (Kadijevich and Haapasalo, 2001). Students' scores in most verbal problems concerning fractions were significantly higher than their scores in these same tasks, which were rewritten directly in the symbolic form at the end of the test (it was not allowed to return to the previous tasks). Thus, MODEM 1993 studies anticipated the findings that in TIMSS 1999 study the Finnish students scored modest in mechanical symbolic tasks but very well in context-oriented tasks of PISA 2000 study. The MODEM -framework also offers promising opportunities to remove the gap between school and university, being a serious concern of ICMI. Ehmke, Pesonen & Haapasalo (2011) found that missing out the orientation and definition phases caused difficulties to university students when trying to utilize dynamical applets to whitewash their naïve and stereotypic conceptions gained in school.

LINKS TO OTHER CHALLENGES

The discussion above shows that for being able to respond to *Ch2* in appropriate way, the other six Challenges must be taken into account in one way or other. The MODEM framework allows educators to respond to *Ch,1* especially in the left hand side of Figure 2 when students make their investigations to find relevant attributes for the concept. Regarding *Ch6*, teaching and assessment should reflect tools, which have been proved to be sustainable in the history of human thinking processes and for the generation of new mathematics. Zimmermann's (2003) study of the history of mathematics reveals eight main activities, which proved to lead very often to new mathematical results at different times and in different cultures for more than 5000 years: *order, find, play, construct, apply, calculate, evaluate,* and *argue*). It might be appropriate to use this framework not only in instructional design but to develop an instrument to measure student profiles: (1) Math-profile: How strong each activity appears when using the term 'mathematics', (2) Identity-profile: How good the student thinks he or she is performing each activity, and (3) Techno-profile: How strongly technology can help to perform each activity.

Studies of Haapasalo & Eronen (2011) suggest that mathematics teaching in school does not give support for those activities, and the support gained from university mathematics seems to be even lower. The only exception is calculating that gets an overdose. On the contrary, doing voluntarily mathematics with a graphic calculator, even during a short period of time outside the classroom, enlarged 8th grade student's mathematical and identity profiles (see Eronen & Haapasalo, 2010). Thus, we come to *Ch3*: *minimalist instruction* and self-regulated learning. Even though for the planning of learning environments a quasi-systematic framework (cf. Figure 2) is crucial, in learning situations students must have freedom to choose the problems that they want to solve within continuous self-evaluation instead of relying on guidance by the teacher. Such an assumption of minimalism, introduced in Carrol (1998) was adopted in the so-called ClassPad Project when students at 8th grade got opportunity to study voluntarily topics of 9thclass mathematics with ClassPad calculator during their summer holiday. This totally strange tool was shortly represented to them just few days before their summer holiday. The only duty was to write a portfolio if they worked with the tool. The sample below is taken from the portfolio of a quite average student (from a 75 minutes work on 15th of July 2005, beginning at 00:27 a clock).



- I draw a line (a). When drag-dropping, the equation is y= **1.613**x-0.5992 (b).
- I change to y= **2**x-0.5992, … the angle between the line and y-axis is getting smaller (c).
- I change to y=**1**x-0.5992, … the angle between the line and y-axis is getting larger.
- I change to y= 1.613x**-0.4**. …I don't see any changes (in the graphic window).
- I change to y= 1.613x**-4**, … the line moves in the same angle away from origin (d).
- I change to y= 1.613x**+4**, …. the line moves in the same way, but to another direction.
- I will continue in the morning. **Time is now 1:42 a.m.. I worked 1 h 15 min.**

Figure 3. Example of instrumentalisation during spontaneous ClassPad work.

The portfolio example reveals that by manipulating the equation spontaneously (i.e ***C***), the student explained how the parameters affect the position and location of the line (i.e. manipulated ***P***). Through *instrumentalisation* she made her own interpretation that the line moves along the x-axis. This interpretation against the standard view appeared also among other students when they six months later studied the whole 9th grade mathematics only with the ClassPad calculator without any textbook or traditional homework. In addition to the fact that the cognitive results were higher than in the traditional teaching, pupils' above-mentioned profiles were extended when measured by using the eight Zimmermann activities (see Eronen & Haapasalo, 2010).

*Ch*3 was responded to by planning the problems for student's investigations within a quasi-systematic MODEM framework (see Figure 2), whilst they were arranged to form a so-called problem buffet where students could "jump the gun" by choosing from this buffet any problem that they wanted. A student team, for example, initially selected a quite complicated problem on optimizing mobile phone costs, which was planned to be a reinforcement task for ***C*** in Figure 1. After realizing that models appeared too difficult for them, they then chose a problem set that happened to consist of identification tasks - the lowest level of understanding the links between representations (cf. Figure 2). This example shows that a sophisticated interplay between a systematic and minimalist approach can be achieved with quite simple pedagogical solutions.

The portfolio example above also demonstrates a response to *Ch4* in the spirit of Haapasalo (2007): "instead of speaking about 'implementing modern technology into classroom' it might be more appropriate to speak about 'adapting mathematics teaching to the needs of information technology in modern society". It also demonstrates a response to *Ch5:* students may be seen as designers of their own lessons, whether *ICT*-based or not, rather than just as knowledge users. The fact that a 14 years old girl was ready to spend 75 minutes to work with ClassPad in the middle of the most beautiful Finnish summer (see Figure 3) shows that mathematics can be "edible and digestible", fulfilling a demand of *Ch7* as suggested in Haapasalo (2008).

REFERENCES

Anderson, J. (1983). *The architecture of cognition.* Cambridge, MA: Harvard University Press.

Ehmke, T., Pesonen, M. & Haapasalo, L. (2011). Assessment of university students' understanding of abstract binary operations. *Nordisk Mathematikdidaktikk* 15 (4), 25-40.

Eronen, L. & Haapasalo, L. (2010). Making Mathematics through Progressive Technology. In B. Sriraman, C. Bergsten, S. Goodchild, G. Palsdottir, B. Dahl, L. Haapasalo (Eds.), *The First Sourcebook on Nordic Research in Mathematics Education* (pp. 701-710). Charlotte, NC: Information Age Publishing.

Carroll, J. M. (Ed.). (1998). *Minimalism beyond the Nurnberg Funnel.* Cambridge, MA: The MIT Press.

Devlin, K. (2007). What is conceptual understanding? *The Mathematical Association of America*. Retrieved from http://www.maa.org/devlin/devlin_09_07.html

Gray, E., & Tall, D. (1993). Success and Failure in Mathematics: The Flexible Meaning of Symbols as Process and Concept. *Mathematics Teaching* 142, 6-10.

Haapasalo, L. (2007). Adapting Mathematics Education to the Needs of ICT. *The Electronic Journal of Mathematics and Technology*, 1 (1), 1-10. Retrieved from https://php.radford.edu/~ejmt/deliveryBoy.php?paper=eJMT_v1n1p1.

Haapasalo, L. (1993). Systematic Constructivism in Mathematical Concept Building. In P. Kupari & L. Haapasalo (Eds.), *Constructivist and Curricular Issues in the*

School Mathematics Education. Publication Series B, Theory and Practice 82 (pp. 9-22). University of Jyväskylä, Institute for Educational Research.

Haapasalo, L. (2008). Perspectives on Instrumental Orchestration and Assessment - from Challenges to Opportunities (plenary). In W.C. Yang (Chair) *The 13th Asian Technology Conference in Mathematics.* Bangkok, Thailand. Retrieved from http://atcm.mathandtech.org/EP2008/papers_invited/2412008_15968.pdf

Haapasalo L.& Eronen L. (2011). Looking Back and Forward on the Light of Survey Studies Related to Mathematics Teacher Education. In H. Silfverberg & J. Joutsenlahti (Eds.) *Integrating Research into Mathematics and Science Education in the 2010s* (pp. 67-84). Tampere: The Finnish Mathematics and Science Education Research Association.

Haapasalo, L. & Kadijevich, Dj. (2000). Two Types of Mathematical Knowledge and Their Relation. *Journal für Mathematik-Didaktik* 21 (2), 139-157.

Haapasalo, L. & Samuels, P. (2011). Responding to the Challenges of Instrumental Orchestration through Physical and Virtual Robotics. *Computers & Education* 57, 1484-1492.

Hiebert, J. & Wearne, D. (1986). Procedures Over Concepts: The Acquisition of Decimal Number Knowledge. In Hiebert, J. (Ed.), *Conceptual and procedural knowledge: the case of mathematics* (pp. 199-223). Hillsdale, NJ: Erlbaum.

Ivic, I. (1989). Profiles of educators: Lev S. Vygotsky. *Prospects* 19 (3), 427-436.

Kadijevich, Dj. & Haapasalo, L. (2001). Linking Procedural and Conceptual Mathematical Knowledge through CAL. *Journal for Computer Assisted Learning* 17, 156-165.

Lauritzen, P. (2012). Conceptual and Procedural Knowledge of Mathematical Functions. *Dissertations in Education, Humanities, and Theology 34*. Joensuu: Publications of the University of Eastern Finland.

Nesher, P. (1986). Are Mathematical Understanding and Algorithmic Performance Related? *For the Learning of Mathematics* 6, 2-9.

Rittle-Johnson, B., Siegler, R. S. & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology* 93 (2), 346-362.

Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics* 14, 44 -55.

Skemp, R. (1987). *The Psychology of Learning Mathematics.* Hillsdale, NJ: Erlbaum.

Star, J. R. (2007). Foregrounding Procedural Knowledge. *Journal for research in Mathematics Education* 38 (2), 132-135

Zimmermann, B. (2003). On the Genesis of Mathematics and Mathematical Thinking - a Network of Motives and Activities drawn from History of Mathematics. In L. Haapasalo & K. Sormunen (Eds.), Towards Meaningful Mathematics and Science Education. *Bulletins of the Faculty of Education 86* (pp. 29-47). Joensuu: University of Joensuu, Faculty of Education.