COMPARING APPROACHES THROUGH A REFERENCE EPISTEMOLOGICAL MODEL: THE CASE OF SCHOOL ALGEBRA

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Luis Radford and Yves Chevallard, whose last two research programmes in mathematics education were awarded with the Hans Freudenthal medal, have both given a prime place to the problem of teaching and learning elementary algebra. However, their approaches are far from being similar. Are they comparable? We are starting a dialogue considering how each approach, in a more or less explicit way, defines what algebra is, that is, how it characterises 'algebraic thinking' or 'algebraic activities' with what we call a reference epistemological model *(REM) of elementary algebra. The dialogue starts by assuming the point of view of the Anthropological Theory of the Didactic (ATD), presenting our own REM, and the kind of questions addressed by this approach, in relation to the Theory of Knowledge Objectification developed by Radford.*

1. TWO WAYS OF APPROACHING SCHOOL ALGEBRA

For more than twenty years, Luis Radford's and Yves Chevallard's investigations have dealt with school algebra as a research domain. Nonetheless, the problems approached present very different formulations and scopes. They seem to deal with completely different worlds and there are very few mutual references, if any. It is clear that we are here considering two approaches that have been 'personalised' by the researchers mentioned but that, as the Hans Freudenthal award states, they represent two research programmes, involving several researchers from different countries. At the same time, the way each programme approaches the research problem of school algebra does not need to be exclusive, even if we are only considering these two. As in a case study, we are considering Luis Radford's approach as a representative of research dealing with 'algebraic thinking' (Radford 2002, 2008, 2012a, 2012b, Radford & Puig 2007). The other case, represented by Yves Chevallard's work, corresponds to the latest investigations carried out within the ATD, mostly by our research team, around what we have called the 'process of algebraization' (Bolea, Bosch & Gascón 1998, 2001, 2004, Ruiz-Munzón 2010, Ruiz-Munzón et al 2012, Bosch 2012).

We will start by briefly formulating some of the main problematic questions or research problems addressed by each of the approaches.

Algebraic thinking within the Theory of Knowledge Objectification

What characterises algebraic thinking? What are the relationships (filiations and ruptures) between numerical or arithmetical and algebraic forms of thinking? Could embodied forms of algebraic thinking observed in adolescents be accessible to young students? What is the evolution of the different components of algebraic thinking in young students? How do students interpret the meaning of algebraic symbolism? What characterises algebraic generalisations and what distinguishes them from arithmetic ones? These are only a few of the many questions that could be formulated in terms of the "iconicity" and "semiotic contraction" in the development of algebraic thinking, the "process of objectification", etc., in the sense these notions adopt in the Theory of Knowledge Objectification (TKO).

The algebraized mathematical activity

In the case of the ATD, the main question is the characterization of "*algebraized*" mathematical activities. It is assumed that the algebraic character of mathematical activity is relative, a question of degree, and some indicators to measure the "*algebraization* degree" of a mathematical praxeology are defined and used to describe the process of *algebraization* of mathematical praxeologies. The questions that can then be posed are, for instance: What conditions are required for elementary algebra to normally exist as a modelling tool in an educational institution (for instance at lower secondary school, grades 7-10) so that the school mathematical organisations can be progressively *algrebraized*? In what sense can lower secondary school mathematical organisations be considered as poorly *algebraized*? What aspects of the *algebraization* process are difficult to introduce at school and what constraints hinder their introduction?

Differences and points in common between both ways of questioning

The first obvious observation is that both approaches question and problematize different aspects of the 'didactic reality' they wish to study: "algebraic thinking" *versus* the "process of *algebraization* of mathematical activities". It may seem that considering problems of such a different nature can make the dialogue between them rather difficult, at least if we stay at the level of the formulation of research problems. In fact, our postulate is that the differences in how problems are formulated are deeply dependent on the way of interpreting and describing algebra in each framework, that is, the *reference epistemological model of school algebra* used. Depending on how we define or consider the mathematical content involved in a didactic problem (school algebra, in our case), we will be able to formulate some research questions rather than others, to delimit the empirical unit of analysis considered, and to look for acceptable answers to these questions. Therefore, the distance between the two ways of approaching the problem of school algebra can be explained by the differences between the epistemological models assumed. We thus propose to initiate the dialogue between both approaches at this level.

2. WHAT ARE REFERENCE EPISTEMOLOGICAL MODELS?

When analysing any teaching or learning process of mathematical contents, questions arise related to the interpretation of the mathematics involved in it. For instance, what is elementary algebra (or geometry, or statistics)? How is it interpreted in a given educational institution? What is it for? How is it related to other contents? Etc. The different institutions interfering in the didactic processes propose more or less explicit answers to said questions. If researchers assume those answers uncritically, they run the risk of not dealing with the empirical facts observed in a sufficiently unbiased way. Therefore, the ATD proposes to elaborate what are called *reference epistemological models* (REM) for the different mathematical sectors or domains involved in teaching and learning processes (Bosch & Gascón 2005). In the ATD, those REM are formulated in terms of local and regional praxeologies and of sequences of linked praxeologies of increasing complexity (Sierra 2006).

It is important to insist on the fact that the epistemological models built by didactic research should be considered as a *working hypotheses*. As such, they are always provisional and constantly need to be contrasted and revised. Even if they are given other names, either more or less explicit, other approaches in mathematics education use analogue theoretical constructs. For instance, the Theory of Didactic Situations proposes to describe mathematical bodies of knowledge in terms of a-didactic or fundamental situations (Brousseau 1997); the APOS theory (Dubinsky & McDonald 2002) uses the “genetic decomposition” of a concept to base its teaching proposals on; the Onto-Semiotic Approach (Godino, Batanero & Font 2006) talks about “systemic configurations”; the theory of “Abstraction in Context” (Dreyfus, Hershkowitz & Schwarz 2001) is concerned with epistemic actions (RBC+C model); etc. In the case here considered, we could say that the Theory of Knowledge Objectification is supported on a model of *algebraic thinking* that proposes a specific way of interpreting and describing elementary algebra.

From the point of view of the Anthropological Theory of the Didactic, the kind of reference epistemological models considered have certain specific features. The empirical data taken into consideration to build them do not only come from school mathematics, but also from the different institutions involved in the process of didactic transposition (the school and its environment, policy-makers, “scholar mathematicians”, professionals, etc.). It is important that REM do not uncritically assume any of the viewpoints that are dominant in these institutions.

In this approach, epistemological models do not take into account the idiosyncrasy of the persons involved in the teaching and learning processes, nor the specific conditions in which they take place. What they explicitly include are the concrete activities that can be considered as the *raison d’être* of the mathematical content involved in terms of problems to be solved or questions to be addressed, as well as the way it takes form and evolves to give rise to new problematic questions.

3. THE ATD REFERENCE EPISTEMOLOGICAL MODEL FOR SCHOOL ALGEBRA

With respect to school algebra, the ATD proposal is to interpret it as a *process of algebraization* of already existing mathematical praxeologies, considering it a *tool* to carry out a modelling activity that ends up affecting all sectors of mathematics. Therefore, algebra does not appear as "one more content" of compulsory mathematics, at the same level as the other mathematical praxeologies learnt at school (like arithmetic, statistics or geometry) but as a general modelling tool of *any* school mathematical praxeology, that is, as a tool to model previously mathematized systems (Bolea, Bosch & Gascón 1998, 2001, 2004; Ruiz-Munzón 2010; Ruiz-Munzón et al 2012). In this interpretation, algebra appears as a practical and theoretical tool, enhancing our power to solve problems, but also as the possibility of questioning, explaining and rearranging already existing bodies of knowledge.

This vision of algebra can provide an answer to the problem of the status and rationale of school algebra in current secondary education. On the one hand, algebra appears as a privileged tool to approach theoretical questions arising in different domains of school mathematics (especially arithmetic and geometry) that cannot be solved within these domains. A well-known example is the work with patterns or sequences where a building principle is given and one needs to make a prediction and then find the rule or general law that characterises it. This feature highlights another differential feature of algebra that is usually referred to as "universal arithmetic": the possibility of using it to study relationships independently of the nature of the related objects, leading to "generalised" solutions of a whole type of problems, instead of a single answer to isolated problems, as is the case in arithmetic. Another essential aspect of the rationale of algebra is the need to organise mathematical tasks in *types* of problems and to introduce the idea of generalisation in the resolution process, a process making full use of letters as parameters.

In this perspective, the introduction of the algebraic tool at school needs to previously have a system to model, that is, a well-known praxeology that could act as a *milieu* (in the sense given to this term in the Theory of Didactic Situations) and that is rich enough to generate, through its modelling, the different entities (algebraic expressions, equations, inequalities, formulae, etc.) essential to the subsequent functioning of the algebraic tool. In the model proposed, this initial system is the set of *calculation programmes* (CP). A CP is a sequence of arithmetic operations applied to an initial set of numbers or quantities that can be effectuated "step by step"- mostly orally and writing the partial results - and provides a final number of quantity as a result. The corpus of problems of classic elementary arithmetic (and also some geometrical ones) can all be solved through the verbal description of a CP and its execution: what was called a "rule" in the old arithmetic books. The starting point of the REM for elementary algebra is therefore a compound of elementary arithmetical praxeologies with techniques based on the verbal description of CP and their effectuation "step by step".

Working with CP soon presents some technical limitations and also raises theoretical questions about, on the one hand, the reasons for obtaining a given result, justifying and interpreting it and, on the other hand, the possible connections between different kinds of problems and techniques. All these questions lead to an enlargement of the initial system through successive modelling processes giving rise to *different stages of the "algebraization"* process that we will briefly summarize hereafter. A more detailed description can be found in (Ruiz-Munzón 2010; Ruiz-Munzón et al 2012).

The *first stage* of the algebraization process starts when it is necessary to consider a CP not only as a process but as a whole, representing it in a "sufficiently material" way—for instance written or graphically—to manipulate it. This does not necessarily mean the use of letters to indicate the different numbers or quantities intervening in a CP (the "variables" or "arguments" of a CP). However, it requires making the global structure of the CP explicit and taking into account the hierarchy of arithmetic operations (the "bracket rules"). This new practice generates the need of new techniques to *create and simplify algebraic expressions* and a new theoretical environment to justify these techniques. It is here where the notions of "algebraic expression"—as the symbolic model of a CP—and of "equivalence" between two CP can be defined. Following the classic terminology about equations, we can say that this stage requires the operation of "simplifying" and "transposing" equivalent terms but not the operation of "cancelling".

The passage to the *second stage* of algebraization occurs when the identity between CP needs to be manipulated. In this stage, algebraic techniques include considering equations (of different degrees) as new mathematical objects, as well as the technical transformations needed to solve them. This case includes the resolution of equations with one unknown and one parameter, that is, the case where problems are modelled with CP with two arguments and the solutions are given as a relationship between the arguments involved. In the specific case where one of the numeric arguments takes on a concrete value, the problem is reduced to solving a one-variable equation. Nowadays, school algebra mainly remains in this last case (without necessarily having passed through the first one): solving one-variable equations of first and second degree and the word problems that can be modelled with these equations, without achieving the second stage of the algebraization process.

The *third stage* of the algebraization process appears when the number of arguments of the CP is not limited and the distinction between unknowns and parameters is eliminated. The new praxeology obtained contains the work of production, transformation and interpretation of *formulae*. It is not much present at current secondary schools even if it appears under a weak form in other disciplines (like physics or chemistry). At least in Spain, the use of algebraic techniques to deal with formulae is hardly disseminated outside the study of the general “linear” and “quadratic” cases. However, they play an essential role in the transition from elementary algebra to functions and differential calculus, a transition that is nowadays quite weakened in school mathematics. Furthermore, secondary school mathematics does not usually include the systematic manipulation of the global structure of the problems approached, which can be reflected in the fact that letters used in algebraic expressions only play the role of unknowns (in equations) or variables (in functions), while parameters are rarely present. However, it can be argued (Chevallard & Bosch 2012) in which sense the omission of parameters—that is, the use of letter to designate “known” as well as “unknown” quantities—can limit the development of efficient modelling algebraic tools and constitutes a clear denaturalisation of the algebraic activity carried out at school.

Let us consider a short example to illustrate the three stages of the process:

> Take a problem of the sort: “Think of a number, multiply it by 4, add 10, divide the result by 2 and subtract the initial number”, a process that we will represent by a CP: $P(n) = (4n + 10)/2 - n$. A problem where we know that $P(n) = 7$ can be solved in the first stage, by first simplifying $P(n)$ and finding the equivalence $P(n) \equiv n + 5$, which gives the result $n = 2$. If the problem is $P(n) = 3n - 7$, the passage to the second stage seems more natural (even if we can always find complex techniques to solve it remaining in the first stage). If the CP is $P(n,a) = (4n + a)/2 - n$ (“Think of a number, multiply by 4, add another number, etc.”) and the problem states that $P(n,a) = 2n - a$, the same type of techniques and theoretical environment enables to find a solution, which here appears as a relationship between n and a. The third stage corresponds to CP with more than 2 arguments, requiring new techniques to describe the relationships obtained, especially when we do not only work with linear equations.

This three-stage model of the algebraization process is a tool to analyse what kind of algebra is taught and learnt in the different educational systems, what elements are left out of the teaching process and what other elements could be integrated under specific conditions to be established. It is complemented with four general *indicators of the degree of algebraization* of a given mathematical content (or praxeology), as proposed by (Bolea, Bosch & Gascón 2001) to analyse the different possible constructions of the algebraic process by looking at the mathematical activities resulting from it. They correspond to: (1) the possibility to manipulate the global structure of the problems (where the systematic use of parameters becomes essential); (2) the need to “objectify” or “thematize” the mathematical techniques used and to question them (with the corresponding emergence of theoretical questioning); (3) the unification and ostensive reduction of praxeologies (their types of problems, techniques and theoretical discourses); (4) the emergence of new problems independent of the modelled systems.

The effort to explicitly state an epistemological reference model for elementary algebra has different purposes. It can first be used as a descriptive tool to analyse the kind of algebraic praxeologies that exist at school and to study the ecological effects (conditions provided and constraints imposed) of these praxeologies in other mathematical contents. It is also a productive tool when trying to connect investigations concerning school algebra carried out from different theoretical perspectives, as it helps specify the reference epistemological model of algebra more or less explicitly assumed by each research and compare the results provided by each one. For instance, one can consider what aspects of elementary algebra are not taught at school and inquire about the possible reasons of their absence, as well as the 'nature' and 'origin' of these reasons (Chevallard & Bosch 2012). Another interesting exploitation would be comparing different research works, as for instance the "structural approach" of the research strand on *Early algebra* or the "algebrafying" paradigm promoted by J. J. Kaput (2000) and the first stage of the algebraization process and its possible implementation in the classroom.

4. THE TKO AND THE PROBLEM OF SCHOOL ALGEBRA

Luis Radford's works present some answers to the questions addressed by the TKO to the problem of school algebra. First of all, *algebraic thinking* is characterized not by the use of symbolism but by its "analytical" character (Radford 2012b, pp. 16-17):

> […] I suggested, on both historical-epistemological and semiotic grounds, that algebraic thinking cannot be reduced to an activity mediated by notations. Although the modern alphanumeric symbolism constitutes a very powerful semiotic system, in no way can it characterize algebraic thinking.
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> Algebraic thinking, I suggested, is rather characterized by the *analytic* manner in which it deals with indeterminate numbers—something where, as two fathers of algebra, Viète (1983) and Descartes (1954), explicitly stated, no difference is made between known and unknown numbers. Looking at algebraic thinking from this perspective opens up new possibilities to rethink the manner in which indeterminate quantities can be signified. It is here where semiotics enters the scene. Indeed, semiotics is interested in understanding the manner in which individuals signify (Eco, 1988).

From this viewpoint, and based on different teaching experiences, algebraic thinking is postulated to emerge early among young students, even if this finding raises new difficulties related to the description of its evolution (Radford 2012b, p. 16):

> From a sensuous perspective on human cognition, it is not difficult to appreciate that 7–8-year-old students can effectively start thinking algebraically. To move to the second research question was much more difficult. How to account for the development of cognitive formations?

When considering algebraic symbolism and students' difficulties with *the interpretation of its meaning*, the operation with the unknown and the use of the first algebraic techniques to solves first degree equations is described in terms of the coordination of different systems of signs (gestures, natural language, drawings, etc.) and the articulation of levels of abstraction. In this case, algebra is considered as a problem-solving tool:

> More precisely, our approach to algebra as a problem-solving tool means the development of an analytic technique based on a conceptually complex kind of mathematical thinking relying on the calculation of known and not-yet-known numbers or magnitudes that acquire a meaning as they are handled in the pursuit of the goal of the activity.
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> [...] EISL [*elementary iconic symbolic language*] soft syntax allowed the students to accomplish the translation of elementary word-problems into an iconic statement and to suitably transform these statements in order to reach the solution. These iconic statements form an iconic text that, in the end, appears as a didactic device reducing the gap between the statement of the problem in natural language

> and a formal symbolic treatment of equations. The didactic goal was not to remove the gap (which is, I believe, an impossible task). The goal was to provide the students with an intermediary semiotic system from where to derive certain meanings to be used later in the semiotic system of symbolic algebra.

Algebraic generalisations are considered fundamental algebraic activities and should be defined in relation to *arithmetical* ones. More generally, the relationship between arithmetic and algebra must be clarified. Here, Radford assumes that the gap is located in the equations of the form $A\ x + B = C\ x + D$ where arithmetic methods (consisting in carrying out inverse operations) fail and the students should learn to operate on the unknown (Filloy & Rojano 1989), thus moving to *analytical* thinking:

> In order to operate on the unknown, or on indeterminate quantities in general (e.g., variables, parameters), one has to think analytically. That is, one has to consider the indeterminate quantities as if they were something known, as if they were specific numbers.

It is considered that, form a genetic point of view, arithmetic differs from algebra in this *analytical thinking* with indeterminate quantities, where unknown and known numbers are treated in the same way. An important consequence of this difference is that *algebraic formulae are deduced*, which is essential to distinguish algebraic generalisations from arithmetic ones. The production of a formula in the generalisation of patterns is not necessarily a sign of algebraic thinking (it can be the result of a guess, for instance). The use of algebraic symbolism is not a condition, neither necessary nor sufficient, to algebraic thinking and this conclusion opens new ways to the study of elementary forms of algebraic thinking in young students.

5. STARTING A DIALOGUE BETWEEN THE TKO AND THE ATD

Our proposal to initiate a dialogue between the TKO and the ATD approaches to school algebra is to start from some questions and assertions formulated within the ATD about how the TKO considers algebra, as a way to compare the *reference epistemological models* proposed by each approach. It is only the very beginning of the dialogue, since it has to be complemented with the reciprocal viewpoint: the vision of the ATD investigations and the TKO's own interpretation of algebra.

In the ATD, the degree of algebraization is a characteristic assigned to mathematical praxeologies as a whole, not to the types of tasks or mathematical techniques considered separately. This point is certainly an important difference regarding the TKO approach. In our opinion, it is necessary to address it by looking more deeply into the general epistemological model *of mathematics* used by the TKO and the way it can be specified in the case of algebra. The relationship between "thinking" and "activity" (or "modes of reflexion" and "action") in the TKO and the notion of "praxeology" in the ATD need to be established more clearly.

The ATD approach agrees with the TKO that a mathematical praxeology can be relatively algebraized (satisfying to a certain extent some of the indicators of the algebraization degree) without the need to explicitly use algebraic symbolism. A clear example is shown in Bolea, Bosch & Gascón (2001) related to "figurate numbers”. Reciprocally, a mathematical praxeology can make extensive use of algebraic symbolism and still remain very poorly algebraized.

There are, however, some issues where the points of view differ and need to be considered more deeply. The first one is related to the tasks dealing with equations of the form Ax + B = D. In the ATD model, these tasks only occupy a small part of the first stage of algebraization and, in the case where simplification techniques are not required, said tasks can even remain in the non-algebraic stage, needing only arithmetical operations to be solved. The same happens with the tasks dealing with equations of the form Ax + B = Cx + D when the solutions obtained are numerical: they are only a small part of the second stage of algebraization. According to the three-stage model, other kinds of equations -or calculation programmes- should be considered in order to let algebraic techniques appear with all their functionality.

The issue of the so-called “analytic character of algebra” that appears to be central in the TKO also plays an important role in the ATD characterization of algebra (see for instance Gascón 1993 and Chevallard 1985, 1989a, 1989b), within the first indicator of the algebraization degree. A question arises about how this “analytic character” can be defined in the TKO when one moves beyond early algebra.

Finally, one of the main functionalities of reference epistemological models is to let researchers get free from the institutional vision of the educational facts they are considering, as has been shown by Bosch (2012) in the case of the ATD approach to algebra. It is important, from this point of view, to consider what aspects of this institutional vision of the teaching and learning of algebra are questioned by the TKO’s proposal and what others are assumed as valid ones.

The dialogue between approaches requires specific work, like the one suggested in this paper, as well as specific tools such as the scrutiny of the epistemological models constructed and assumed by each frame. How is the “mathematical content” delimited? What is it made of? How is it related to the “scholarly” and school vision of the content? What phenomena does it enable to highlight or explain? What kind of teaching intervention does it suggest?

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