# ON THE GEOMETRICAL MEANINGS OF MULTIPLICATION: GEOMETRICAL WORK SPACE, SEMIOTIC MEDIATION AND STUDENTS' CHOSEN PATHS

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*In this article, I briefly present a combination of different theoretical approaches – Mathematical Work Space, Registers of Semiotic Representation and Semiotic Mediation Theory – in order to analyze students' paths within an experimental lesson connecting multiplication and some of its geometric meanings. I will also present reasons for combining these theories and finally I will illustrate how they have been used to analyze results from our experiments conducted in French high schools.*

## FROM INITIAL QUESTIONS TO A THEORETICAL COMBINATION FOR ANALYZING STUDENTS' CHOSEN PATHS

### Geometry as a link between multiplication and its meanings for different sets of numbers

The fact that the notion of multiplication is closely associated with the idea of calculation can impede students from imagining a geometrical representation of the product. In the same way, the association between real numbers and the notion of magnitude can also get in the way when representing negative numbers, as well as when giving meaning to the product of two negative numbers. Thus, the multiplication of negative integers does not allow a geometric representation unless the "quantities" are treated in terms of orientation and direction (Argand, 1806). The extension of the operations' definition for complex numbers is linked to the representation of imaginary quantities by vectors. As a result, transformation is the only context in which multiplication and some of its geometric meanings can be connected. We are establishing a relationship between the different meanings of our mathematical object, with geometry as the glue holding them all together.

Geometric representations encourage the use of cognitive variables favoring the understanding of a mathematical object. The abstraction of arithmetic and algebraic concepts also stems from the fact that they are only represented through a symbolic diagram (Radford, 2003) where a sign "bears an arbitrary or non-motivated relationship to its signified" (Radford, 2003, p.5). So, in addition to these signs, it seems to us that it is always necessary to have an intermediary between a conception and access to its meanings, given that "there is not mathematical thinking without using semiotic representations" (Duval, 2008, p.1).

Consequently, we can formulate a key question related to the empirical context of our research: will students be able to establish connections between multiplication and geometry? We will see how we have integrated these three elements – multiplication, its meanings and geometric transformations – in an original experimental situation, which has been created to respond to this question. This is the question that will be specifically analyzed using the combination of theories presented below.

## Our main theoretical framework: The Geometrical Work Space

After analyzing the notion of Mathematical Work Space (MWS) (Kuzniak, 2011), we determined that this theoretical approach could suitably account for the complexity and richness of students' mathematical work. This notion assumes that a network has been created on two levels, one cognitive and the other epistemological. This network relies on a certain number of *geneses*, which can be *semiotic*, *instrumental* or *discursive* (cf. Figure 1). The analysis of this bilateral relation allowed us to expand our theoretical knowledge of mathematical work spaces and to determine the theory's flexibility. As we will see, this flexibility allows us to combine several theories in order to analyze students' chosen paths within an empirical mathematical working space.

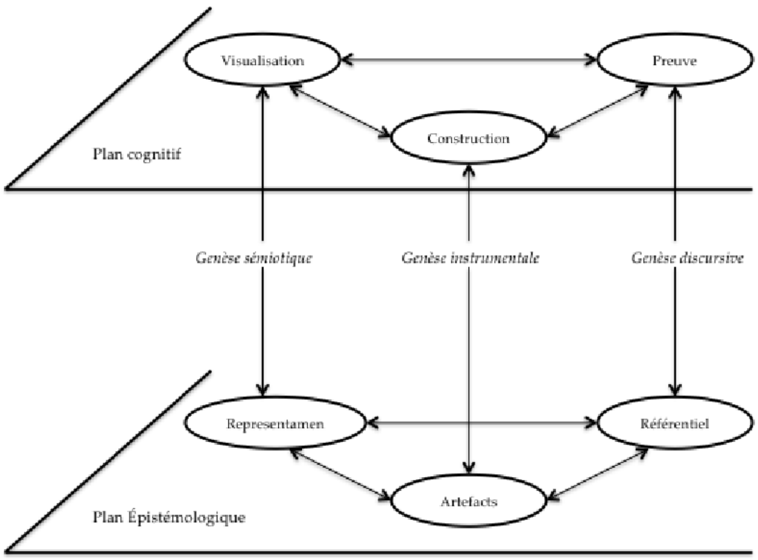



**Figure 1: A genetic approach to the Geometrical Work Space**

The starting point for the *geneses* linking the two levels of the MWS is traditionally placed on the epistemological level: for example, the visualization of an abstract mathematical object in a real or material space can be produced by the manipulation of artifacts in the construction of a figure. Still, the components on the epistemological level can be set in motion by needs on the cognitive level. A construction with artifacts can respond to a need for demonstration; the construction of a figure in a *paper-pencil* environment can be the result of a visualization allowing certain properties to assume a new configuration, or it can assemble the elements necessary for a proof. Thus, within these processes, called *geneses*, we can see not only the existence but also the permanent interactions between different registers of semiotic representation: we can make the transition from proof to construction through a change in register of representation (cognitive entrance); a geometric configuration, a sign or *representamen* (epistemological entrance) can prompt a visualization (cognitive action) making use of the properties and axioms (epistemolocial action) leading to a proof. As Duval states, “the only way to have access to [mathematical objects] is using signs or semiotic representations” (Duval, 2006, p. 107).

Finally, we have the makings of a hypothesis: being conscious of the metaphorical meaning of a mathematical object could allow a point of entry starting at the cognitive level of an MWS, which at the same time would encourage manipulating the components of the epistemological level. The metaphorical characteristics of this mathematical object, whose meaning we wish to construct using geometry, would therefore allow students to establish the transition between the cognitive level and the components of the epistemological level:

> “Metaphors are not just rhetorical devices, but powerful cognitive tools that help us to build or grasp new concepts, as well as solving problems in efficient and friendly ways” (Soto-Andrade & Reyes-Santander, 2011, p. 2).

## The combination of theories: Mathematical Work Space (MWS) and the role of sign-artifacts in a socially interactive space

All of the different studies dealing with semiotic notions present in the process of learning/teaching mathematics—whether they include information technology or not, whether or not they talk about registers of semiotic representation or pay special attention to the role of language and the understanding of mathematical objects—all of these positions “[revolve] around the relationship between mathematics and semiotics, concern questions of an epistemological, cognitive and sociocultural order” (Falcade, 2006, p. 3-4). However, within an MWS, the didactic question does not include explicit interactions between different individuals. Additionally, the MWS does not necessarily include other intermediaries, aside from the teacher, between the learners and the knowledge to be acquired or developed. It seems appropriate, then, to explicitly include *mediator intermediaries* where mathematics and semiotics can be found, where the different *geneses*, figural, discursive and instrumental, occur, at the point where *semiotic mediation* and, when possible, *social mediation*, can facilitate access to research and the acquisition of meaning of mathematical objects. That said, given our special interest for semiotic genesis within a mathematical work space, the limitations of the existing semiotic approach as well as the theoretical definition of artifacts (Kuzniak, 2004) led us to look for other theoretical approaches dealing with semiotic mediation and the social construction of mathematical knowledge. We concentrated on Bartolini Bussi and

Mariotti's (2008) work on Semiotic Mediation Theory, aspects of which we associated with Radford (2004) and Sfard's (2008) reflections on the social construction of mathematical knowledge and the complexity of the process of understanding a mathematical object.

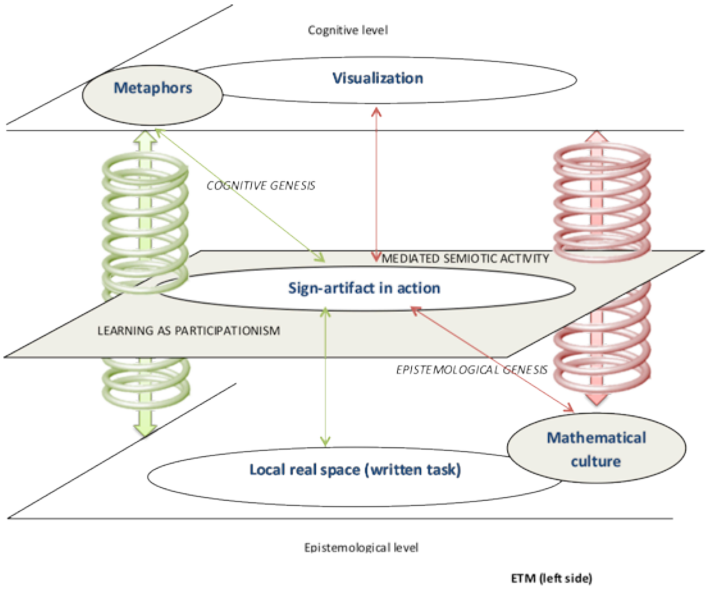



**Figure 1: Diagram showing the dynamic aspect of the MWS's components and the arrangement of the epistemological and cognitive levels, caused by the action of the sign-artifact in a context of semiotic mediation.**

From a didactic point of view, Semiotic Mediation Theory includes elements such as the direct manipulation of tools, either in the form of concrete objects taken from the history of mathematics, or in the form of technological artifacts. The theory also considers the precise organization of work in the classroom, where the relationships between the individual dimension, work in pairs and the collective dimension all play a role, and where oral and written activities complement one another. Finally, the theory also considers students' reading and interpretation of historical primary sources, aided by the teacher (Falcade, 2006).

The inclusion of historical and/or technological mediators as sign-artifacts on the one hand, and on the other hand the importance of collaborative work within the learning-teaching process, were the key elements that brought us to integrate Semiotic Mediation Theory and the Mathematical Work Space. Thus, we've included the MWS in a socio-constructivist learning process where the sociocultural and semiotic dimensions are included in the proximal development zone defined by Vygotsky (1934-1997).

## AN OUTLINE OF OUR METHODOLOGY

Our desire to study the understanding of multiplication in a geometric context led us to design experimental course material. Observing several students' work on this non-traditional material allowed us to study their ways of solving problems in a mathematics lesson requiring changes of register of semiotic representation in a process of semiotic mediation.

Students in *Terminale S* (twelfth grade scientific track) were asked to solve a series of five questions suggesting a geometric approach to the multiplication of real and complex numbers (Appendix 1)[1].The activity was introduced in four *Terminale S* classes by their teachers. Thirty-four groups of two to four students worked on the activity for two hours in class. This session was integrated into the usual series of lessons by the teachers, who had just begun a chapter on complex numbers. At first, the students were instructed to make a geometric construction of the product of two real numbers in the plane, as proposed by Descartes in his *Geometry* (1637). Next, the students had to find a relationship between the points given on a plane and the multiplication of complex numbers. The final question of the series[2] called on students to think back on the entire activity. It played a fundamental role in the exploratory process, and its analysis allowed a first description of the paths followed by students moving between Descartes' multiplication and the understanding of the geometric meanings of multiplication for different sets of numbers. In order to describe the role of geometrization in students' approaches to multiplication, we studied their way of solving geometric construction problems involving the multiplication of real and complex numbers. Gradually, we've began forming a response to our research question (cf. Introducing the mathematical content of analysis: linking multiplication to geometry) "now transformed" and seeing it through the eyes of our main theoretical framework: are there interactions between the cognitive and epistemological levels of the MWS employed by students showing evidence of a geometric understanding of multiplication? Through this methodology, we determined students' chosen paths between Descartes' multiplication and the understanding of the geometric meanings of multiplication for different sets of numbers. In this article, we outline two paths so as to illustrate some of the experiment's results and the way we used our combination of theories to analyze students' mathematical work. In this work we analyzed the use of previously studied mathematical content and the way students employ it; interactions produced between the components of the MWS employed by students; the role played, as a *sign-artifact*, by the configuration of Thales' theorem corresponding to the geometric representation of Descartes' multiplication; the identification the *origin of geneses in a geometrical work space*. The mediating sign-artifact is therefore an essential element of our didactic proposals. As we will see, our artifact is a sign: a mathematical sign, a geometric representation and an icon of Thales' theorem. Recognized by students in the first question of the series as a tool to be employed in a proof, the sign must evolve throughout the collaborative lesson. The goal of its systematic use in the activities is the collective production of new signs, which correspond to new interpretations of the same artifact. We can associate this last point, especially concerning the evolution of signs and the way they influence discourse, with what Anna Sfard calls a "visual mediation." This is the place where “visual mediators have been defined as providers of the images with which discursants identify the object of their talk and coordinate their communication” (Sfard, 2008, p.147).We hope that our theoretical combination will allow us to study whether geometric representations, either given to or produced by the student, recognized as psychological tools or *sign-artifacts*, are capable of producing a precise mathematical object—in this case, multiplication.

## STUDENTS’ PATHS: ANALYZING A FEW RESULTS THROUGH THE EYES OF OUR THEORETICAL COMBINATION

The individuals (groups) taking part in our experiment were initially classified by hand (i.e. we studied each sequence and each response to the final question, looking for elements of an answer that either corresponded to or differed from our initial determination) according to their responses to the last question of the series, leading to a first classification with three possible types of responses: Transformation (T); Proportionality and Thales’ theorem (PTTh); complex (C), without explicit references to geometric transformations. The determination of students’ paths was thus based on an analysis of the entire process leading them to their response to the last question.

### Comparison between two groups showing different conclusions: Proportionality and Thales’ theorem (PTTh) and Transformation (T).

The two groups were initially given two different classifications. We will examine the groups’ differences beyond certain similarities in their responses to the final question. Group C1-I9 (T) bases its conclusion on an immediate visual connection between multiplication and the sign rule as seen in the Cartesian plane. Then the group extends this relation to Descartes’ multiplication as well as any multiplication with any type of factor. The most striking observation we can make about their response, and which shows the close links between figural and discursive geneses, is the connection made between the sign rule and vectors’ angles. The change of frames related to the sign rule is due to our activity, because the geometric manifestation of the sign rule does not appear in the French curriculum. In order to reinforce this idea, we emphasize that the group specifically identified the nature of the angles, especially the zero angle and the flat angle which allow a connection between real and complex numbers in this geometric configuration.

Group C3-I1 (PTTh) arrives at a conclusion that takes into account the different parts of the lesson. They show the properties of multiplication of real and complex numbers, justifying them with Thales’ theorem.

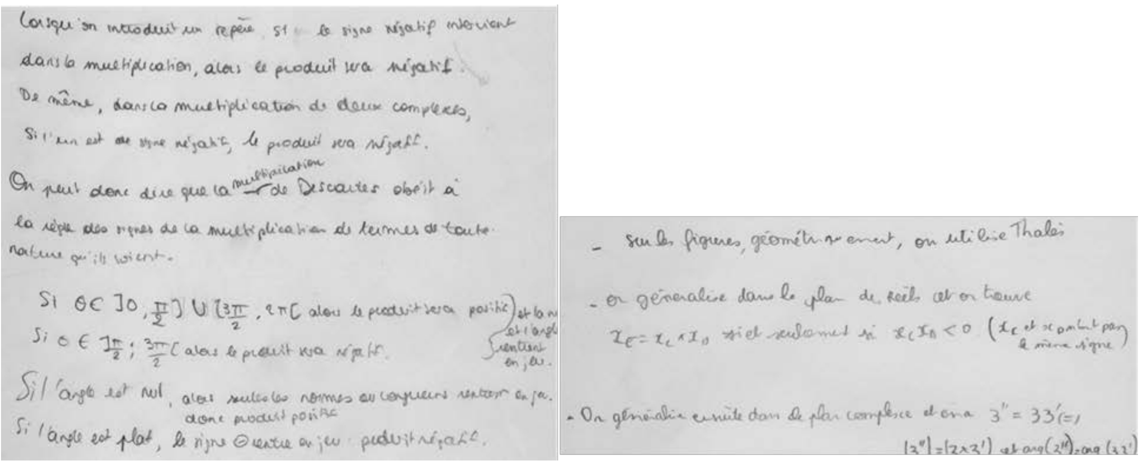



**Figure 4: Left, pair C1-I9's conclusion (T). Right, pair C3-I1's conclusion (PTTh)**

They present their geometric interpretation of multiplication of real and complex numbers as a generalization of Thales' theorem in the Cartesian plane and then in the complex plane. Can we say that the word "generalization" clearly accounts for a connection between the different aspects of the lesson? In a way, yes, because the lesson requires students to extend different sets of numbers.

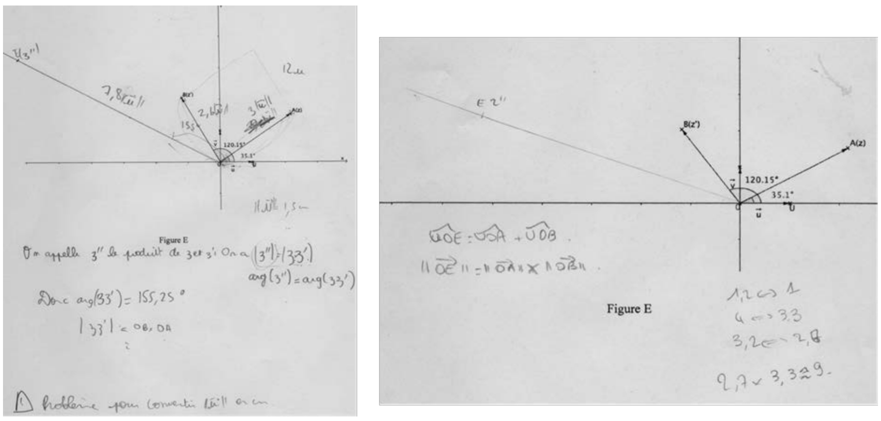
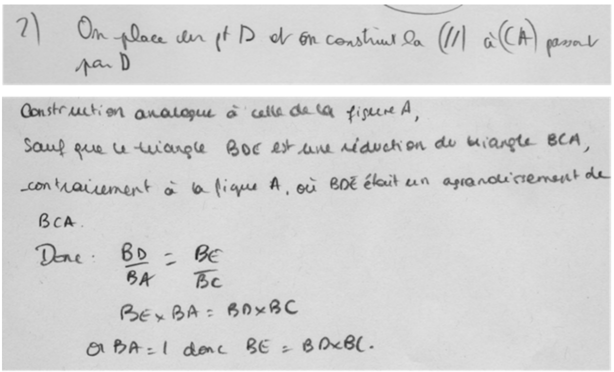



**Figure 5: Left, C3-I1's answer to 4.b (PTTh). Right, C1-I9's response to 4.b (T)**

For a more complete view, the figures above show the complete responses of both groups to question 4.b. It seems clear that the groups' algebraic knowledge of complex numbers guided their geometric construction, which shows no connection with the lesson's previous constructions. The algebraic properties of the multiplication of complex numbers are already part of students' *theoretical references* and they orient the students' construction, which is correct but isolated from the rest of the lesson. In their responses, nothing explicitly accounts for the visualization and geometric comprehension of complex numbers as a transformation in the plane. The entrance into the MWS for this question is epistemological, then, resulting in a *construction* with a ruler allowing the students to visualize the placement of the product of two complex numbers in the plane. In this lesson, the two groups take different approaches to question 2b. Here, only group C1-I9 (T) has used *transformations* in its response to one of the activity's first questions.


2) On place un pt D et on construit la (//) à (CA) passant par D

Construction analogue à celle de la figure A,
Sauf que le triangle BDE est une réduction du triangle BCA,
contrairement à la figure A, où BDE était un agrandissement de BCA.

Donc : $\frac{BD}{BA} = \frac{BE}{BC}$

$BE \times BA = BD \times BC$

Or $BA = 1$ donc $BE = BD \times BC$.

**Figure 6: Above, group C3-I1's response to question 2.b. Below, response 2.b for group C1-I9.**

In analyzing this response, it is quite interesting that the students refer to the *reduction* and *enlargement* of triangle BCA. The relationship of proportionality implied in Thales' Theorem and the geometric representation of Descartes' multiplication were not approached according to the segments-factors and the segment-product representing the proportionality. The visualization of similar triangles favors the immediate use of Thales' theorem because the students interpret this reduction-enlargement using their already available knowledge. These *paths* present their own specific characteristics. The "T" group mentions transformations quite early, in the second question, and their conclusion (final response) is very rich, relating the sign rule and geometry. The group using Thales produces a response describing multiplication for different sets of numbers but does not explicitly link the icon of Thales to a geometric representation of multiplication for different sets of numbers.

**Synthesis of the analysis**

| **Variable** | **Analysis results** |
|---|---|
| **Entry into the MWS** | Mixed but largely epistemological. |
| **Semiotic genesis/links between different registers of representation** | Semiotic genesis of unknown origin, especially in the response given by C1-I9 to the second question. A semiotic genesis of cognitive origin may have occurred with the visualization of similar triangles, followed by the visualization of a transformation (reduction-enlargement) of these triangles through multiplication. A significant link between registers of representation was made during the association between the sign rule and the representation of the product of a positive and a negative number in the "affine" plane (C3-I1). |
| **Semiotic mediation of the sign-** | The action and evolution of the *sign-artifact* were identified thanks to a specific explanation of the existence of a zero angle in Descartes' product (which was necessarily transposed into the |

| | |
|---|---|
| **artifact** | "affine" plane). A link was therefore produced between the properties of the icon of Thales' theorem and the properties of multiplication of complex numbers. |
| **Geometrical meaning of multiplication** | Hypothetically, the product was interpreted as resulting from a transformation in the plane. This could have been a possible interpretation of the sign rule in terms of angles and by generalizing the meaning of any product to the product of two complex numbers. |

## Conclusion

We based our theoretical framework on a unified conception of cognitive and didactic elements. Several interests informed its development: the social dimension of learning processes; the study of semiotic mediation processes favoring the collaborative construction of a mathematical object; and the construction of meaning of mathematical objects. New lines of questioning emerge as a result of the diversity of concluding responses and the similarities and differences between the different students' chosen *paths*. They also demonstrate the difficulty of organizing the different *geneses* of the MWS. We realize that this is quite a complex cognitive activity since there is no direct conversion between one register of representation and another. This leads us to position the students' activity within a mathematical work space where the meaning of mathematical objects *emerges* as a result of a *cognitive genesis*. This *genesis* assumes the presence of complex semiotic interactions, such as those described by D'Amore and Fandino (2007) in order to describe the difficulties of moving between different representations. For example, the transposition of Descartes' product to a product operating directly on numbers, represented geometrically, positioned on a plane: this can only be the result of realizing that the activity is based on a *mathematical idea* (Lakoff & Nunez, 1997) that completely departs from the traditional knowledge of the mathematical object in question, i.e. the geometric meanings of multiplication. We are no longer working on the techniques of calculation or proof. Thus, because of the richness of the MWS introduced by the teacher as well as the diversity of students' personal MWSs, we must highlight the importance of our theoretical combination (MWS (Kuzniak, 2011) and TSM (Bartolini Bussi & Mariotti, 2008)). Through the lens of these theories, we have seen and analyzed students' ways of looking for the meanings of a mathematical object, dealing with a mathematical sign-artifact between an epistemological and a cognitive level, within a context of social interactions.

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[1] Link to the series of questions: https://docs.google.com/open?id=0B2PIBsYMh2gCS2EwQzYxSFlUSkk

[2] Last question: “Thinking about the work you have done today and in past mathematics lessons, what geometric meaning could you assign to multiplication?”