THRESHOLD CONSTRUCTS INSTRUMENTING TEACHERS' ORCHESTRATION OF AN INQUIRY WITH GEOGEBRA

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In this paper we investigate how to efficiently empower teachers to implement and orchestrate a mathematical learning activity supported by digital technologies. The particular learning activity in this study is intended to facilitate learners' transition from the Pythagorean Theorem to the distance formula and the equation of a circle. The activity comprises structured and guided inquiries involving laptops with GeoGebra and traditional resources and has been tested with 38 upper secondary students and two mathematics teachers. Our results indicate that a singular discussion with the teachers, based on the researcher's prospective analysis of the activity with main focus on threshold constructs and self-regulation, suffices to support the teachers' implementation and orchestration of the activity.

INTRODUCTION

There is not only an abundance of digital technologies available in society, but also an abundance of research about learning mathematics with technologies. Still, research and current teaching practices do not seem to provide sufficient guidance on how to efficiently and systematically integrate digital technologies in mathematics education (Drijvers, 2012). However, numerous efforts in design-based research provide examples of singular good practices that serve as inspiration and proofs of existence that technologies can significantly enhance the teaching and learning of mathematics (Hegedus and Moreno-Armella, 2009; Sollervall and Milrad, 2012; Drijvers, 2012). The issue of scaling up innovations is being increasingly attended to by the mathematics education research community (e.g., Jo Boaler, plenary session at PME-NA 2012; Hegedus and Lesh, 2008). Several authors have recognized that research needs to address systemic issues at macro-level as well as how to efficiently implement classroom innovations at micro-level (e.g., Hegedus and Lesh, 2008).

In the current effort, we will address the implementation of a mathematical learning activity at micro-level, involving two teachers and 38 students in the beginning of their second year at the Natural Sciences Programme in upper secondary school.

Moving beyond teacher-driven improvement of their current practices, as addressed by Lesson Studies and Learning Studies (Lo, Marton, Pang, and Pong, 2004), we have in several research efforts applied a research-driven approach where researchers with complementing domains of expertise engage in the collaborative design of innovative mathematical learning activities supported by digital technologies

(Sollervall and Milrad, 2012). In order to fully exploit the opportunities for learning that are afforded by digital technologies, we have chosen to give the participating researchers with technological expertise be responsible for communicating the affordances of these technologies to the members of the research team. Rather than involving teachers at the level of technologies, we have designed prototypical activities that are presented to and adjusted by the teachers before implementation (Sollervall and Milrad, 2012). Because of the technical complexity in some of these prototypical activities, we have engaged expertise also in the implementation phase.

In the current research effort, we are making use of a stable, commonly used, and readily accessible interactive software, namely GeoGebra, that allows the teachers to be in charge of implementing the activity with their students. The three tasks in the activity are presented and discussed during a meeting on the day before implementation, with focus on mathematical constructs and possible teacher interventions that may be crucial for the students' successful completion of the tasks. The teachers propose additional possible obstacles that are discussed and addressed through alterations of the activity. Within the limited scope of this paper, it is impossible to address all aspects of the learning activity in detail. Instead, we will present the activity in a similar way that it was presented to the teachers, although this implies a superficial treatment of some of the theoretical underpinnings for the activity.

The activity blends students' constructions in GeoGebra and traditional work with pen and paper. The development of the activity is framed by the methodology of design-based research that allows us to attend not only to traditional outcome evaluation but also development of learning activities and prospective (a priori) analysis of hypothetical outcomes (Cobb, Confrey, diSessa, Lehrer and Schauble, 2003). The development phase involves negotiations of a preliminary activity with a prospective analysis based on hypothetical learning trajectories (HLT: Drijvers, 2003; Sollervall and Milrad, 2012). Another key aspect of design-based research is its cyclic character that allows adjustment and improvement of an activity (Drijvers, 2003). The version presented in this paper is in its second iteration. The first version was implemented with three first-year secondary students in May, 2012. The evaluation of this implementation resulted in minor alterations of the technical instructions and a decision to involve laptops instead of an interactive whiteboard. The latter decision was mainly due to an ambition to implement the second iteration of the activity in a whole class setting.

Furthermore, a restructuring was made by sequencing the first two tasks according to the involved mathematical processes – defining, representing, generalizing, and justifying – and the corresponding actions on the products of these processes, following the Processes and Actions framework (Zbiek, Heid, and Blume, 2012). These four processes, as well as other theoretical constructs, were used in our discussions with the teachers. In this paper, we investigate the issues addressed in these discussions and how they influence the implementation of the activity.

RESEARCH OBJECTIVES AND RESEARCH QUESTIONS

This paper addresses the implementation of an inquiry-based learning activity, intended to facilitate mathematical processes and actions related to upper secondary students' transition from the Pythagorean Theorem to the distance formula and the equation of a circle. As the students engage in the activity, they encounter specific threshold constructs – as external (physical) or internal (mental) constructs – that serve a crucial role in promoting their continued mathematical inquiry.

Our research questions are:

- How are threshold constructs and self-regulating skills addressed in the discussions between the researcher and the teachers?
- How do the problems that students encounter during the implemented activity relate to the threshold constructs and self-regulating skills?

In the next section, we discuss the notion of threshold construct as a local version of threshold concepts (Meyer and Land, 2005). We also address the notion of inquiry and the self-regulating skills that facilitate successfully completing an inquiry.

THEORETICAL AND METHODOLOGICAL CONSIDERATIONS

From a teacher perspective, it is essential to identify and address the threshold concepts that facilitate learning in a specific subject area. An example is the threshold concept 'limit' in Calculus. In general, threshold concepts can be seen as 'conceptual gateways' that lead to previously inaccessible ways of thinking about something (Meyer and Land, 2005). In this study, we will attend to threshold concepts that are localized to the learning trajectories for a specific activity focusing on learning coordinate geometry and specifically the distance formula. We refer to these local and specific threshold concepts as *threshold constructs* for the activity. These threshold constructs can also be interpreted as epistemological obstacles (Sierpinska, 1987) that may obstruct a student's learning trajectory. For example, given the task to find an algebraic expression for the distance between A = (0,4) and P = (x,y) in Figure 2, the triangle PAC serves as a threshold construct (when the distance formula is not yet available) and affords attending to the point C and its coordinates, which in turn afford constructing algebraic expressions for the catheti of PAC so that the Pythagorean Theorem can be applied to complete the task. Such threshold constructs may be identified either in a prospective analysis as hypothetical constructs or actual constructs that during the implemented activity serve a crucial role in facilitating a student's learning trajectory. Since the threshold constructs are defined from a student learning perspective, they need to be empirically tested, updated and refined in an iterative research process that can be smoothly integrated within the methodology of design-based research.

The activity that will be presented in the next section is comprised of structured and guided inquiries (Herron, 1971) where the students have primary ownership and initiative. Inquiry-based learning challenges the students' self-regulation regarding cognition, motivation, behavior, and context, in corresponding phases of self-regulation: forethought, planning, and activation (cognition); monitoring (motivation); control (behavior); reaction and reflection (context) (Schunk, 2005).

In this paper, we will investigate if (and, if so, how) the teachers' interventions address scaffolding these phases of self-regulation or if they address other issues, with focus on the threshold constructs.

THE ACTIVITY AS PRESENTED TO THE TEACHERS

A meeting between the researcher and the two teachers (and a third teacher) took place in the afternoon on August 29, 2012, the day before implementation. Before the meeting, the teachers had received a preliminary printed version of the student instructions. These instructions consisted of a cover page with general information, seven pages with task instructions, and a brief two page manual to GeoGebra.

The first two tasks were presented as structured inquiries, while the third task was a less structured so called guided inquiry with no suggested sequence of steps for solving the task (Herron, 1971). The teachers were informed that self-regulated inquiries impose a substantial cognitive load on the students, which implies a need for the students to consolidate their experiences in order to process and retain what they have just learned (Kirschner, Sweller, and Clark, 2006). We decided to address this issue by allowing time (15 minutes) for the students to write individual reflections at the end of the activity. The teachers agreed to follow up on these reflections during the following lesson. This arrangement left 2.5 hours (8.30 am – 11.00 am) for working with the three tasks, including a short break.

As the activity involves interaction with laptops supporting the software GeoGebra that the students have only used on a few previous occasions, the issue of instrumental genesis (Verillion and Rabardel, 1995) was specifically addressed in the design process. We designed the first two tasks as staging activities involving not only mathematical learning objectives but also serving to familiarize the students with the character of the activity and specifically the involved technologies (Edelson, Gordin, and Pea, 1999). Furthermore, it was agreed to give the students sufficient time to explore, investigate and resolve problems by themselves, and that guidance should be given restrictively, although this could imply that they will not manage to complete all the tasks. It was agreed to be sufficient if the students complete two of the three tasks.

The activity was designed for students who are familiar with the Pythagorean Theorem, the square of a binomial, and the equation of a straight line, but with no previous experience of working with the distance formula. The general objective for the activity was to let students explore geometric relationships, pose hypotheses, confirm these hypotheses by producing algebraic proofs, and thus making connections between geometric and algebraic representations.

In the first task, the students are asked to find (other) points equidistant to two given points P and Q, that is, with the ratio 1:1 between the distances (Fig. 1).

This first task involves working only with GeoGebra following instructions that begin with setting **Perspectives** to **Basic Geometry**, thus giving the students a clear background to work on, showing no grid and no coordinate axes in order to stimulate them in attending only to the geometric relations. The students are asked to first place two points P and Q beside each other on the screen. Then they are asked to place a third point in the screen and were given the following information:

The condition in this task is that the new point should be located equally far away from the points P and Q:

The distance from the point to P should be equal to the distance from the point to Q.

One of the teachers objected that one of these sentences could be removed since they both give the same information. The researcher argued that the redundancy could help students who may be able to interpret one of the sentences but not the other. This position was further supported by arguing that the sentences are qualitatively different, as the first sentence addresses the condition from a procedural point of view while the second sentence objectifies the condition. It was decided to not change the phrasing of the sentences but to keep them as in the proposal.

Next, the students are informed that there are many points that satisfy the condition. They are instructed to place several such points on the screen (such a possible construction is illustrated in Fig. 1, left pane) and then answer the following question:

What geometric figure do you get from all the points that satisfy the condition?

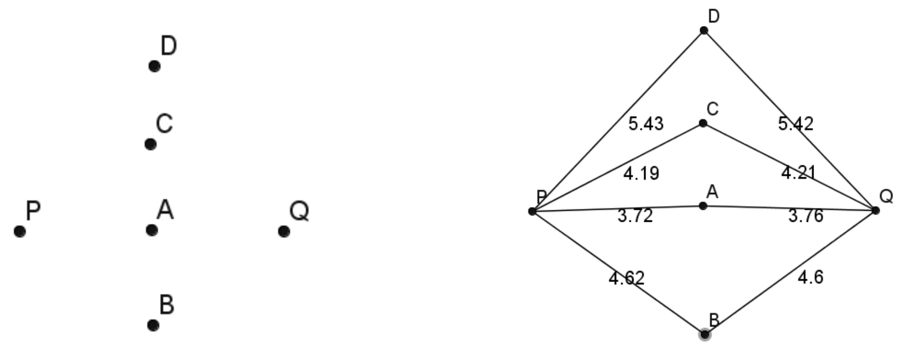

Figure 1: Constructions in GeoGebra according to the instructions for the first task.

On the next page, the students are invited to check the placement of their points, and if necessary adjust their placement, by having GeoGebra measure the distances from each point to P and Q, respectively (Fig. 1, right pane).

The second task addresses the same geometric 1:1 condition and the same question, but now with the points P and Q placed in a coordinate system. The students are instructed to keep their constructs from the first task, show **grid** and **axes**, and place the points at P = (0,4) and Q = (2,0) respectively. A construction according to the geometric 1:1 condition is illustrated in Figure 2 (right pane).

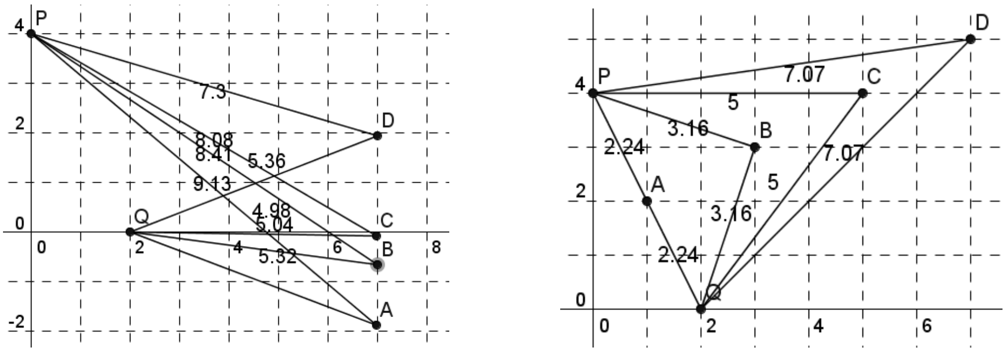

Figure 2: A possible initial setup (left pane) and a correct construction (right pane).

On the next page of instructions the students are asked to work with pen and paper to find the equation of the straight line.

So far, the tasks have consisted of following instructions. In relation to the Processes and Actions framework (Zbiek et al., 2012) the students have been acting on definitions by interpreting and representing them, generalizing from a few points to all points, justifying geometric constructs by using GeoGebra, and acting on a generalization of a visual representation by representing it algebraically.

The continuation of the second task is more challenging. The students are now asked to *prove algebraically that all the points (x,y) that satisfy the condition lie on a straight line*. They are first instructed to remove all points except P, Q, A, and remove the labels on the segments AP and AQ. Next, they are asked to change the labels for P and Q so that they show **Name & Value**. Thereafter, they are asked to place the point A upwards and to the right of P and Q, and place a new point B directly to the right of P and directly below A (Fig. 3, left pane).

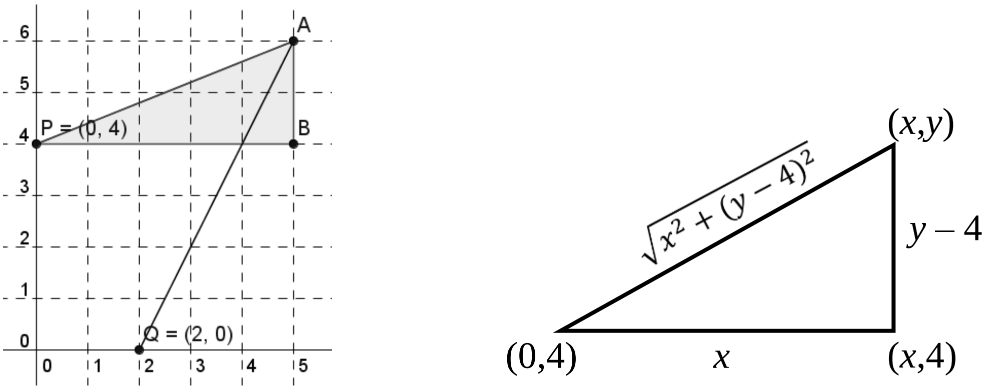

Figure 3: Continuation of task 2 in Geogebra (left pane) and on paper (right pane).

The students are then asked to draw the triangle PAB in GeoGebra (Fig. 3, left pane). Before turning to the next page of instructions, where they are instructed to work with pen and paper, the students are asked how the hypotenuse of the triangle relates to the task they are working on.

They are asked to copy the triangle PAB onto a piece of paper, including the coordinates for P and B. They are asked to set A = (x,y). Their first subtask is to find algebraic expressions for the catheti of the triangle, and the second subtask is to find an algebraic expression for the hypotenuse. In the first iteration, the students were instructed to first determine the coordinates for the point B, but in the discussions with the teachers it was decided to leave this construct as a challenge to the students. Next, the students were asked to find an algebraic expression for the distance between A and Q, without sequencing any steps in the instructions.

Finally, the students were instructed to set the lengths of AP and AQ equal to each other and simplify the equation. As expected, many students did not simplify carefully and made numerous algebraic errors. However, quite a few groups of students managed the first two tasks more or less on their own. Only one group of three students made a serious attempt on the third task, which we do not account for here. Instead, we proceed to highlight some of the presented results in relation to the research questions. We address both questions under each heading.

DISCRIMINATING AND GENERALIZING VISUALIZED POINTS

Already during the first iteration of the activity, it was confirmed that the reference points P and Q (compare Fig. 1) interfered with the interpretation of the geometric figure. The students' first guess was 'a rhombus'. During the second iteration, several groups initially answered 'a cross' or 'a triangle', even after they had constructed the line segments that allowed them to measure distances. Our interpretation is that the picture (Fig. 1) was prioritized before the condition, although the written condition was emphasized in the instructions by being italicized in a large font size. A comment by one of the teachers 'they are not used to working with figures as sets of points' can be seen as further explaining why the picture was favored in the students' work. Interventions by the teachers (and the researcher) involved suggestions to test each point against the condition by reading the condition for each specific point. When the students finally read 'the distance from P to P' they quickly concluded that P does not belong to the figure. Furthermore, the teachers pointed out to several groups that the figure consists of all the points that satisfy the condition, not only the points they have placed on the screen. These incidents relate to the fundamental psychological processes of discrimination and generalization (Bruner, 1966) that the students are expected to apply in relation to a mathematical definition. Discriminating a geometric figure in a picture, generalizing a geometric figure from a finite set of points, and justifying by acting on a formal definition, can be regarded as threshold concepts for our activity.

CONTROL AND REFLECTION EXPLAIN ALGEBRAIC OBSTACLES

As already mentioned, several students made incorrect treatments of the algebraic expressions in the final part of task 2, for example the classical mistakes of 1) replacing the square of a binomial with the sum of the squares of the two terms and 2) canceling square roots and squares term by term in the equation

$$\sqrt{(x-0)^2+(y-4)^2}=\sqrt{(x-2)^2+(y-0)^2}$$

Many students worked very quickly with this subtask. When asked to check their expressions, they readily found their mistakes to example 1 by writing the square as the product of two binomials. Example 2 was mainly handled by recall. Our conclusion is that they knew the algebraic rules, but proceeded carelessly by working too quickly and not taking care in structuring their written expressions. Hence, the algebraic rules should not be regarded as threshold constructs for these students. Their algebraic mistakes can instead be explained by insufficient self-regulating skills related to control, as the mistakes are caused by the behavior of working carelessly, and reflection, which is particularly important in the inquiry context.

A SCAFFOLDED THRESHOLD CONCEPT BECOMES A CHALLENGE

During the first iteration, the students were instructed to determine the coordinates for the point B in Figure 3. This instruction can be interpreted as an embedded scaffolding feature that affords finding algebraic expressions for the catheti. As mentioned earlier, it was decided to remove this particular instruction in the second iteration and instead relying on the teachers to provide guidance to the students, if needed. Instead of providing embedded scaffolding for the particular threshold construct B = $(x,4)$, it was transformed into a genuine challenge for the students. Some of the students managed this particular task by themselves, while those who got stuck were suggested by the teacher to attend to the coordinates of B. This went smoothly, as the teachers were aware of this particular threshold construct.

In design-based research, particularly with technologies, the aim is often to work in the opposite direction and attempt to replace teacher scaffolding actions with scaffolds that are embedded in the activity, thus reducing the demands on the teacher and enhancing the possibilities for a teacher accepting to integrate the activity in her classroom (Wong, Looi, Boticki, and Sun, 2011). On the other hand, it is desirable to empower the students, show trust in their capabilities, and thus nurturing their collaborative and self-regulating skills (ibid.). Our approach of negotiating threshold constructs and issues of self-regulation with the teachers allows us to achieve a reasonable balance between teacher scaffolding actions, embedded scaffolds, and challenges intended to empower the students in their collaborative work.

CONCLUDING REMARKS

In a classroom environment where students mainly solve problems in a textbook, most mistakes can be easily adjusted by checking answers or asking peers or the teacher. In such an environment, individual students are not responsible for monitoring, control, and reflection on their work. Mistakes do not propagate and do not affect future work, so students are not stimulated to develop strategies for self-regulation. In comparison, even a minor mistake in an inquiry can have fatal effect on its continuation and may cause the students to fail in achieving the intended learning objectives. The open inquiry is sometimes, but certainly not always, the best format for an inquiry. By initially planning a structured inquiry we expose its inherent challenges regarding threshold constructs and demands for self-regulation. Negotiating implementation of the activity with teachers creates awareness of these challenges and demands. This process empowers the teachers to take informed decisions about restructuring and possibly open up the inquiry by re-distributing scaffolds on the teacher and peers or embedding them in the activity. The negotiation phase also contributes to empower the teacher as orchestrator of the comprehensive activity and its scaffolds. In our opinion, this is an efficient way to implement research-designed activities in the authentic setting of the teacher's own classroom. From a research perspective, the negotiation phase is a good opportunity to bring teachers' knowledge of content in relation to students and teaching into the research process (KCS and KCT: Ball, Thames and Phelps, 2008).

It is apparent that GeoGebra works well in the upper secondary school classroom as a stable and user-friendly software with commands that align well with mathematical thinking and notation. Minor adjustments of settings are readily handled and specific commands are shared among the students. However, the pedagogical implementation of GeoGebra in the mathematics classroom has to be carefully considered so that it not only supports solving the tasks but also supports the students in achieving the mathematical learning objectives. In our activity, it would have been easier to have GeoGebra provide the equation of the straight line in the second task. Instead, we chose to let the students do this tedious work by hand, particularly since the learning objective was the distance formula and not the equation of the straight line.

Furthermore, it may be noted that the distance formula was not explicitly addressed in our activity, but rather emerged as a useful tool for solving a task in coordinate geometry. The teachers commented on this particular aspect by saying 'this is really a good way to work with coordinate geometry, instead of working with the dull standardized tasks in the textbook'. We wish that this may imply that the students who engaged in our activity have learned to appreciate the distance formula as a good thing to know and not only yet another formula to memorize.

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