TEACHING FUNCTIONS AT TERTIARY LEVEL – WHAT IS MADE POSSIBLE TO LEARN?

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The study reported in the present paper forms part of an ongoing project regarding the teaching of functions in undergraduate courses at three Swedish universities. The theoretical framework underlying the project is Sfard's commognitive theory. In the present study, three lectures in calculus from two different universities are analyzed, focusing on the opportunities for learning regarding the function concept afforded by the discursive practices of the teachers. All three teachers are found to make extensive use of the principle of variation, but differences are found for instance in the emphasis put on different realizations of functions. Also, according to the type of content of the lectures, differences between more process-oriented and more object-oriented discourses of functions can be seen.

Key words: Functions, Discourse, Teaching, Tertiary mathematics, Commognitive theory

INTRODUCTION

The teaching of mathematics at tertiary level has attracted growing interest within the research community over the last decade. However, studies of the actual teaching practices of university mathematics teachers remain relatively rare. My thesis project, focusing on the teaching practices of university teachers regarding the function concept, contributes to this area of research. The first part of the project, reports of which have been presented at PME 35 (Viirman 2011) and ICME 12 (Viirman 2012), aims at describing and categorising the discursive practices of the teachers regarding functions. The second part, of which this paper is a first report, focuses on possibilities for learning. The aim of the present paper is investigating what possible learning regarding the function concept is afforded by the discursive practices of the teachers.

THEORETICAL FRAMEWORK

Underlying the project is the view of mathematics as a discursive activity (e.g. Dörfler 2002, p. 339). According to this view, mathematical objects are discursively constituted, and doing mathematics is engaging in mathematical discourse. Central to this approach is the commognitive theory of Sfard (2008). Taking the assumption "that *patterned, collective forms of distinctly human forms of doing are developmentally prior to the activities of the individual*" (ibid, p. 78, emph. in original) as a starting point, Sfard defines thinking as "an individualized version of (interpersonal) communicating" (ibid, p. 81). The notion of communication is thus central to commognitive theory, and it is used to define what Sfard means by discourse. Different types of communication are called discourses, and they can be distinguished through four characteristics: *word use*, *visual mediators*, *narratives* (sequences of utterances regarding objects and their relations, subject to endorsement or rejection within the discourse) and *routines* (repetitive patterns characteristic of the discourse) (ibid, pp. 133-135). The rules of discourse act on two different levels. Object-level rules regard the properties of the objects of the discourse, while meta-level rules govern the actions of the discursants. A routine can thus be seen as a set of meta-rules describing a repetitive discursive action (ibid, p. 208). Within the commognitive framework, learning is viewed as change in the individualized discourse. Learning can then take place on two levels, related to the levels of rules of discourse. Object-level learning is expressed through expansion of existing discourse, whereas meta-level learning involves changes in the meta-rules of the discourse (ibid, pp. 255-256). Central to meta-level learning is the notion of commognitive conflict, a term used to describe a situation where different discursants act according to different meta-rules. Most often, such conflict can be detected through the fact that different discursants endorse contradicting narratives.

However, if student participation is mostly implicit, which is often the case in university mathematics teaching, detecting commognitive conflict can be very hard. Thus some other tool for investigating the possibilities of learning is needed. Runesson (2005) (see also Marton, Runesson & Tsui 2004; Mason 2008) describe learning in terms of becoming aware of dimensions of variation in the critical aspects of the object to be learnt. She claims “that exposure to variation is critical for the possibility to learn, and that what is learned reflects the pattern of variation that was present in the learning situation.” (Runesson 2005, p. 72) Since these patterns of variation are made visible through patterns in the teaching discourse, that is, in the routines of the discourse, the notion of patterns of variation fits within the commognitive framework.

Thus, the question the study aims to answer is the following: What possibilities for learning regarding the function concept, on both object- and meta-level, are afforded by the discursive practices of the teachers?

PREVIOUS RESEARCH

The function concept has been widely studied within mathematics education research (see e.g. Harel & Dubinsky 1992; Romberg, Fennema & Carpenter 1993). Much research has focused on student difficulties with the concept. Of relevance to the present paper is for instance Even (1998), illustrating how knowledge of different representations of functions is dependent on several other factors, for instance the context of the problem, and the ability to consider both the global and point-wise behavior of a function. Sfard’s (1991) notion of process-object duality, while not directly related to the function concept, has still been highly influential on later research on the topic. Research (e.g. Breidenbach, Dubinsky, Hawks & Nichols

1992; Viirman, Attorps & Tossavainen 2010) has shown that students mostly possess process conceptions of function, and that attaining a structural conception is very difficult for many students. The process-object duality is a critical aspect of meta-level learning regarding the function concept, as is the notion of arbitrariness (e.g. Even 1990) associated with a structural view of functions. Critical object-level aspects of the function concept highlighted in the literature are for instance the role of domain and range and defining rule in the definition (e.g. Breidenbach et al 1992; Vinner & Dreyfus 1989), the importance of one-valuedness (e.g. Even 1990), and the different representations of functions – e.g. algebraic, graphic and numerical (e.g. Even 1998; Schwarz & Dreyfus 1995).

The teaching of mathematics at university level, while nowhere near as well-researched as the function concept, has still grown as a research field in later years, for instance through the work of Jaworski and Nardi (e.g. Iannone & Nardi 2005; Nardi, Jaworski & Hegedus 2005), studying university mathematicians' views on the teaching of mathematics. Other studies of particular relevance for the present paper are for instance Pritchard (2010) and Rodd (2003), arguing the continued relevance of lectures in university mathematics teaching, as well as Weber, Porter & Housman (2008), discussing the different roles examples play in conceptual understanding. There are also a number of studies focusing on the actual teaching practices of university mathematics teachers (e.g. Weber 2004; Wood, Joyce, Petocz & Rodd 2007). In Sweden, research on teaching at tertiary level is still rare. Apart from my own work (Viirman 2011; 2012) on the discursive practices of university mathematics teachers, one example is Bergsten (2007), using a case study of one calculus lecture on limits of functions to discuss aspects of quality in mathematics lectures.

METHOD

The empirical data in my project consists of video recordings of lectures and lessons given by teachers in first year mathematics courses at three Swedish universities – one large, internationally renowned; one more recently established; and one smaller, regional university. The participating teachers were chosen among those willing to participate, and giving relevant courses during the time available for data collection. A total of seven teachers participated in the study – four from the large university (labelled A1-A4), two from the younger (B1-B2) and one from the regional university (C1). The topics taught by the teachers included calculus, introductory algebra and linear algebra. For the purposes of the present paper, I have chosen to focus on the lectures in calculus, limiting the number of teachers to three – A1, A4 and B1. Of these teachers, one is female (A1) and two male. All three are quite experienced teachers, having taught at tertiary level for more than 20 years. Two have doctoral degrees in mathematics, while the third (B1) was educated as an upper secondary school teacher. In all three cases, the courses were aimed at engineering students. The topic of teacher A1 is an introduction to the function concept, while teacher A4 covers continuity and teacher B1 the inverse trigonometric functions and their derivatives.

For each of the three teachers, I have about two hours of videotaped lectures, which have been transcribed verbatim, speech as well as the writing on the board. The transcribed lectures were analyzed, with the aim of distinguishing the discursive activities characterizing the teachers' respective discourses of functions, paying special attention to repetitive patterns (for more details, see Viirman 2012) and also patterns of variation regarding the critical aspects of the function concept mentioned in the previous section. I first analyzed each lecture separately, and then compared them, searching for differences and similarities. I have intentionally chosen an outsider perspective, trying to view the unfolding discourse in as unbiased a way as possible. At the same time, I am of course making use of the fact that my mathematical knowledge makes me an insider to the discourse. However, I have specifically tried to avoid making references to what is *not* present in the discourse, except in contrasting the teachers' discursive activities, or making comparisons with previous research.

RESULTS

Looking at the discursive practices of the three teachers in this study, we find that all three teachers use various patterns of variation, highlighting different aspects of the function concept. For instance, the examples used by teacher A1 when introducing the function concept make the various components of the definition visible through variation. The examples she uses are $f(x)=(x-1)^2+2$, $g(x)=\sqrt{x}$ and $h(x)=\sqrt{|x|}$, varying rule, domain and range, as well as indicating the connection between them. She then uses the example $x^2+y^2=1$, using this as contrast, making the one-valuedness requirement visible. As seen in the following excerpt, she even uses the same defining formula while restricting the domain, to highlight the fact that all three parts of the definition are necessary (all excerpts are translated from Swedish by the author. Text within [square brackets] indicates writing on the board):

Teacher A1 [On the board two graphs are drawn: the function $f(x)=x^2$ defined on the intervals [-1, 1] and [-1, 2] respectively.]

And what I want to get at is that this proper definition of what a function is.

It is important that you state both the sets and the rule, not just the rule, really, because these are of course different functions, you can just look at them.

Later, when introducing trigonometric functions, she investigates the behaviour of the functions through variation of angles, periods and amplitudes. We can note, however, that all these functions are real-valued functions of one real variable, making the notion of arbitrariness of domain and range impossible to discern. In fact, with one exception, all examples of functions given by the three teachers are of this type. But since all data are taken from courses in single-variable calculus, where the general concept of function is not included in the syllabus, this is not very surprising.

Teacher B1, when introducing the inverse trigonometric functions, highlights the requirements for the existence of an inverse by varying the functions, as well as their domains, showing how too large a domain results in loss of injectivity, and how different restrictions in domain need to be made for each function. Later, having defined the inverse cosine function, he highlights the relation between cosine and its inverse by varying angles. He asks for the value of cosine at $\pi/3$, $3\pi/2$ and $5\pi/3$, and then for the value of arccos(1/2), to which a student promptly answers $5\pi/3$, giving the teacher the opportunity to illustrate, by means of the unit circle, how the periodicity of the cosine function helps you to get back within the range of the inverse. However, he doesn't take the opportunity to develop this, through further variation of angles. Instead he chooses to focus on calculating the inverse cosine at various points, possibly leaving the students with the feeling that this case is somehow exceptional.

Teacher A4, finally, in a lecture on continuity, uses variation in function graphs to make the possible types of discontinuity visible. He draws a graph which is nice and continuous up to a certain point, marked by a dotted vertical line. He then says:

Teacher A4	Let's assume for example that the function is reasonable in this way, it might go there and then approach some line here. What can happen when we approach this line from the other direction? What possibilities do we have? And they turn out not to be very many.

He then asks the students to give suggestions, and together with them he constructs a catalogue of possibilities. He introduces further variation through describing the different cases both geometrically and analytically, through limits. Later, when discussing one of the basic theorems about continuous functions, the boundedness theorem, he highlights the necessity of the conditions (continuous function on a closed, bounded interval) by removing them one at a time and presenting counterexamples.

Regarding different realizations [1] of functions, for teacher A1 the algebraic realization of a function through a formula is given prominence. New functions are introduced as formulas, while for instance graphs are spoken of as pictures of the function, as in this typical example:

Teacher A1	[$f(x) = (x-1)^2 + 2$] It is a function; it is the function x-squared that I move one step to the right and two steps upwards. Then we can for example draw it.

This pattern occurs nearly every time an example of a function is given. It can be found also in the practices of the other two teachers, but not to the same extent. Also, for teacher A1, the graph is rarely used to gain knowledge of the function. For instance, when determining the range of the function in the excerpt above, she uses the formula to determine the minimum, instead of using the graph. For teacher B1, the formula is still central, but the graphical realization is not just seen as a picture, but as a means of gaining information about the function, as in the following excerpt (he has just drawn the graph of sin(*x*) on the board):

Teacher B1 You sense that we get a problem here, I think.

Because you can't exactly say that this is everywhere increasing, nor that it is everywhere decreasing, and it simply must, it can't have much of an inverse, this one, because if we choose one *y* here.

[He marks a point on the *y*-axis, a short distance above the origin, and draws a line through the point parallel to the *x*-axis.]

There will be a whole lot of possible *x*s, right?

The relationship between graph and formula is also used, for instance to find the graph of the inverse given the graph of the function. Teacher A4, finally, is less reliant on the algebraic presentation of function. He often introduces functions through graphs, as in the example of the classification of discontinuities described above. Of course he also uses formulas, but his more varied approach is likely to help students see the connections between different realizations. He also speaks of "the graph of the function", indicating that the graph is a realization in its own right, whereas for instance teacher A1 mainly speaks in terms of "what does this function look like?", giving the impression that the graph is just an illustration.

On the meta-level, the move from viewing functions as processes on mathematical objects, to viewing them as mathematical objects in their own right, is known to be problematic for students. This duality is seen in various ways in the discourses of the teachers, and the distinction is not always made clear. For instance, all three teachers speak of functions as objects, which can for instance grow, be moved around or be split into smaller parts:

Teacher A1 It is the function x-squared which I move one step to the right and two steps upwards.

Teacher B1 There was a word collecting functions which are either just increasing increasing increasing or just decreasing decreasing decreasing.

Teacher A4 This function really approaches a whole lot of different points here, right.

Teacher A4 We have this function, this kind of patched-together function that consists of two pieces, and what I'm asking is: can you give a value for *b* such that *f* becomes continuous?

In their working with functions, however, distinct differences in the discursive practices of the teachers can be seen. Teacher A1 speaks of functions as objects performing processes on numbers:

Teacher A1 Well, it was all right putting in all real numbers here, all real numbers we can subtract one and take the square and then add two, and what comes out are also real numbers.

Teacher A1 Well, you can say that *f* itself is the machine, and *A* is the set of things you are allowed to put into the machine, and when you put a thing from *A* into the machine, then something comes out that is in *B.*

In the teaching of teacher A4, the transition from process to object is taken a step further. He often makes no distinction between the function and its values, but still doesn't treat the image as a totality, instead referring to it using metaphors of moving, like in the following example (concerning drawing the graph of the function $f(x)=\frac{1}{x-1}$):

Teacher A4 What happens to this function when x is bigger than one?

(...)

It goes down, yes, and then it will wander here, and become bigger and bigger and bigger, and when we approach one, this one is still positive and really big.

This way of handling functions is also apparent in the example referred to above, about the classification of discontinuities:

Teacher A4 In what way can it go wrong when I come wandering here towards this line?

One way is that I hit the line, there is a limit but it is the wrong limit. Is there something else?

Student Oscillation.

Teacher A4 Yes, it could swing, right, so that we don't have a limit when we approach here, that is, that it starts getting the shivers here.

For teacher B1, finally, functions are treated very much like objects, where the global behaviour of the function affects invertibility, and where you can create new functions from old for instance by restricting the domain, or by differentiation. He still speaks of the value of a function at a point, but the function as a process acting on numbers is no longer so prominent.

DISCUSSION

As mentioned above, the lectures of the teachers in the study cover different topics – the definition of function (teacher A1), continuity (teacher A4) and derivatives of the inverse trigonometric functions (teacher B1).These topics typically occur at different stages of a calculus course, with functions defined early, continuity introduced somewhat later, and derivatives of inverse functions covered even later. This fits with the move towards a more objectified discourse observed in the teaching of the three teachers. Since I have no way of accessing what has occurred outside of the video-taped lectures, I can merely make conjectures about the effects of this. To validate these, I would have to observe the same teacher at different stages in the same course. However, it seems to me as if there is room for commognitive conflict here, with potential for meta-level learning for the students. Teacher A1 speaks of a function as an object performing a process, acting upon numbers. For teacher A4, however, the word 'function' signifies something slightly different. This can be seen in the way he sometimes conflates the function with its values, indicating a view of the function as a totality, comprising domain, range and rule. This description is given by teacher A1, but it isn't apparent in her discursive practices the way it is for teacher A4. However, there is a slight tension in his lecture, seen for instance in the way he uses metaphors of movement, indicating a process view. It is known from previous research (Sfard 1991; Breidenbach et al 1992) that attaining a structural view of the function concept is difficult, and that such a view is rare even among university mathematics students (Viirman, Attorps & Tossavainen 2010). Making this tension more explicit might help the students change their own meta-rules.

Concerning object-level learning, research has shown that the use of multiple representations contributes to students' conceptual understanding, and that the ability to move between different representations is dependent on other factors, like the use of a global or point-wise approach (Even 1998) or the use of different kinds of language (Wood et al 2007). In the teaching observed in the present study, the two main realizations of functions are formulas (algebraic) and graphs (geometric). The relation between these is made more or less explicit in the teaching, with teacher A1 mainly treating the graph as an illustration of the formula, and teacher A4 using graphs independently of formulas, to investigate the qualitative behaviour of functions. Also, teacher A1 mainly takes a point-wise view of functions, whereas teachers A4 and B1 use both point-wise and global approaches. It would appear as if teacher A1 is less successful in integrating the use of algebraic and geometric realizations, giving prominence to the algebraic language of formulas.

I also wish to discuss some observations regarding the use of examples. Weber, Porter & Housman (2008) consider two types of example usage in university mathematics teaching – worked examples and examples aimed at concept image building (see also Pritchard 2010). The teachers in the present study display both types in their teaching, but perhaps most interesting to discuss here is the use of examples to develop discursive objects through variation of critical features (Runesson 2005). The way teacher A1 introduces the function concept through a series of examples (and non-examples), and the way teacher A4 develops the notion of discontinuity at a point through a systematic use of examples are cases in point.

Finally, I want to mention one topic of further study suggested by my data. This paper has focused on the function concept, but a very interesting aspect of the discursive practices of the teachers in the study concerns how their views of mathematics as a more general practice are made visible. The scope of this paper is too limited for me to be able to present such an analysis here, but this is planned as the topic of a future publication.

1. Within the commognitive framework, Sfard (2008, p. 155) uses the term ‘realization’ rather than ‘representation’, signifying the fact that the realization and the signifier being realized belong to the same ontological category.

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