CONCEPTUAL UNDERSTANDING IN LINEAR ALGEBRA

- RECONSTRUCTION OF MATHEMATICS STUDENTS' MENTAL STRUCTURES OF THE CONCEPT 'BASIS' -

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Abstract. *In this paper I present my theoretical tool for reconstructing students' mental structures and my application of the tool in qualitative data analyses. The tool is based upon Peirce's semiotics. The data is taken from clinical interviews, in which mathematics students were given tasks dealing with the concept of basis. The aim is to reconstruct students' individual mental structures concerning this concept.*

Keywords: mathematics students, linear algebra, triad of Peirce, mental structure, conceptual understanding

AIM OF THE PAPER

The theory of linear algebra is characterised by its high degree of interconnections between concepts. The concept of basis, for example, is directly connected to the concept of linear independence and the concept of a spanning set. Both are linked as defining conditions to basis. These concepts refer to other concepts – linear combination and span – so that one can with good reason state that basis is a concept of higher order. Understanding the concept of basis implies that one has a high ability to connect ideas.

The research questions of this paper are as follows: Which individual mental structures of the concept basis of a mathematics student can be reconstructed when students solve a particular task concerning this topic? Additionally, this leads to the question: How can a person's mental structure be reconstructed empirically? According to this, I demonstrate my theoretical tool for analysing students' mental structures. I apply the tool via examples to thinking processes of students who are working on a task. Then the reconstructed thinking processes are condensed in a net by focusing on the individual's mental structure of the concept of basis. Before doing so, I give a literature review of previous studies dealing with students' difficulties in advanced mathematics.

STUDENTS' DIFFICULTIES IN LINEAR ALGEBRA

Several studies are dealing with students' difficulties of the conceptual nature of linear algebra. Sierpinska (2000) states that concepts are not used precisely and are not well-conceived. Moreover, students struggle to interpret signs in definitions and to use them for the construction of contextual mental structures. Several researchers of the MAA-Notes (Carlson *et al.*, 1997) report that their students are able to reproduce procedures in known contexts successfully, but that they often fail to understand the meaning behind the procedures. Stewart and Thomas (2010) tie their study in with this aspect by trying out a framework for teaching the concept of basis (and other concepts) that emphasises the embodied aspect of basis. Their aim was to help students enrich their understanding. “The results of this study show that a number of the students tended to prefer to work procedurally“ (p. 186). Maracci (2008) shows that students conceived of linear combinations more as processes than as objects (in terms of Sfard’s process-object duality). Britton and Henderson (2009) described students’ difficulties in dealing with the concept of closure. Students struggled with applying the formal concept of a vector space to the algebraic notation in a task. Furthermore, many of the difficulties that students experience in linear algebra are presented in detail in Dorier et al. (2000). These studies all highlight that students struggle with the conceptual nature of the strictly formalised and abstract theory of linear algebra.

MENTAL STRUCTURES AND UNDERSTANDING

Several authors describe levels of learners’ construction of concepts (hierarchical or otherwise) (e.g. Vollrath, 1984; Winter, 1983; Harel, 1997). All have in common that it is essential to connect single ideas for understanding a concept. This corresponds to psychological views (e.g. Skemp, 1976; Sweller, 2006). According to Skemp (1976) pairing single ideas with concepts, by connecting them, results in a construction of a new idea, which he calls a “relation” (p. 37). A “transformation” (ibid., p. 37) is applied to an idea and describes a function of the idea. The entire process of connecting and transforming ideas results in a construction of a complex structure. “The study of structures themselves is an important part of mathematics; and the study of the ways in which they are built up, and function, is at the very core of the psychology of learning mathematics” (ibid., p. 39).

A mental structure is known as “schema” (ibid., p. 39). It is the “major instrument (…) for solving new problems” (ibid., p. 43) and for the formation of concepts that are yet new. Based on mental structures, Skemp defines understanding something as being able to assimilate it into an appropriate mental schema. Understanding is subjective. I assume that an individual’s mental structure has to be actively constructed by the individual. Information given to him or her in the form of external representations needs to be constructed by the individual. In mathematics, such a construction of mental structures takes place in dealing with signs. Thus, I combine the cognitivistic constructivism approach with the semiotic approach.

THEORETICAL TOOL – ASPECTS OF PEIRCE’S SEMIOTICS

The triadic sign model of Charles Sanders Peirce (1931-1935) is taken as a foundation for reconstructing thinking processes and mental structures by using an interpretative approach. The sign model of Peirce has been applied successfully by some researchers in mathematics education (e.g. Hoffmann, 2003; Presmeg, 2006; Schreiber, 2012; Bikner-Ahsbahs, 2006).

In the next section, I will show how Peirce’s triadic model can serve as a basis for a reconstruction of thinking processes and mental structures.

The triadic sign model of Peirce

Reconstruction means to interpret students' interpretations that are constructed and applied while working on a task. An important aspect of a student's thinking process is the process of interpreting signs. In order to structure a student's process of interpreting, Peirce's sign model is used. Peirce describes the triadic sign model in the following way:

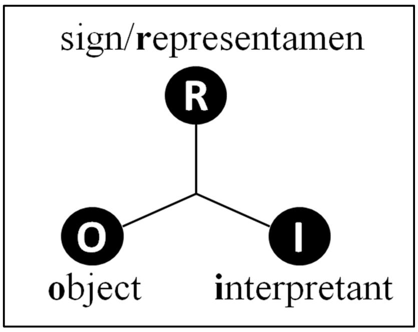

Figure 1: Triad of Peirce

"A sign, or representamen, is something which stands to somebody for something in some respect (…). It addresses somebody, that is, creates in the mind of that person an equivalent sign or perhaps a more developed sign. That sign which it creates I call the interpretant of the first sign. The sign stands for something, its object. It stands for this object not in all respects, but in reference to a sort of idea, which I have sometimes called the ground of representamen" (Peirce, CP 2.228).

According to Peirce, a sign is an external representation that is visible, hearable, or perceptible in some way. The interpretant describes an internal representation that is induced by the sign and can become explicit. Concerning the term object, Peirce differentiates between the 'immediate' and the 'dynamic object' (Hoffmann, 2005). The immediate object is to be understood as the object to which the interpretant of the sign refers. Thus, the immediate object describes a special view towards the sign. "Immediate objects are neither obvious nor visible, but they can be reconstructed (as hypotheses) and verified through further data" (Bikner-Ahsbahs, 2006, p. 163). Unlike the immediate object, the dynamic object would emerge after all possible interpretations of its sign. These interpretations do not depend on individual views. Thus, the dynamic object is kind of a limit-object.

The frame as a foundation of Peirce's triad

Hoffmann (2005) describes the difference between a person working on a familiar sign and the same person working on a sign yet new to him. While interpreting a new sign, the focus is on the sign itself as problematic. While working on a familiar sign, persons "conflate them with the phenomenon itself" (Roth and Bowen, 2003, p. 468 as cited in Hoffmann, 2005, p. 35), as they possess "experience" (Hoffmann, 2005, p. 39) concerning the sign.

Schreiber (2012) does not only use the term 'experience', but explains that the interpretant in Peirce's triad is "determined by the concepts, theories, habits, and skills of the observer" (p. 7), "which are given mentally or physically" (p. 7). This is what Peirce defines as 'ground of representamen' or 'idea' (see quotation of Peirce above). Mentally represented concepts and theories imply the way in which single ideas of a concept or a theory refer to each other. It is in this sense that Skemp's approach of mental structures can be linked to Peirce's theory. Peirce's ground of representamen includes the mental structure of an individual. It influences individual construction of interpretants because “each individual creates interpretants against the background of his or her own subjective interpretation experiences and under a specific perspective” (Schreiber, 2012, p. 7). The interpretants depend on the individual’s ground of representamen.

The chaining of single triads

The strength of Peirce’s triadic model lies in describing processes in which new ideas are constructed. This construction is an ongoing process in which an interpretant of a sign becomes explicit and serves in turn as a sign of a further triad (Peirce, CP 2.303). Presmeg (2006) called this process “chaining” (p. 169).

In the processes that I have reconstructed, the interpretants often do not only refer to the previous sign, but to groups of triads that previously occurred.

METHODOLOGICAL TOOL

Sample and data collection

The sample of the whole study consists of 15 mathematics students, whereby in this paper I only look at two mathematics student, Peter and Mike, to exemplify the analysing tool. Peter and Mike took part in the lectures and tutorials of the linear algebra course in 2010/11, but had to take part again in 2011/12 (their third semester) because they failed the final examination. They participated voluntarily in an additional workshop (four sessions, main topics: vector spaces, basis). The lectures, tutorials, and workshop were taught by different teachers with individual ways of teaching. This is why Peter and Mike received various presentations of basis.

The data was collected about four weeks after the final examination in 2012. This time was chosen because I was interested in the concepts Peter and Mike were able to retain for a longer period of time. In a face to face situation Peter was given a task in written form (see fig. 2). He had unlimited time to work on this task by himself. So, this was an individual and active process of construction in which Peter shows on which conceptual aspects he focuses. The interviewer did not influence this part. After working on this task Peter was asked to explain his procedure to the interviewer. This was a kind of retrospective think aloud. In this part, Peter reflects on his results and investigates their viability. Moreover, the interview offers the possibility of validating the written data. The same procedure was carried out with the student Mike (and the other 13 students).

Ambiguities were expected because students could keep their written argumentation very short or express themselves unclearly. This was experienced in a pilot study. Thus, the interviewer’s role is to act warily in asking and to contain herself. The intention is not to arrive at a solution as quickly as possible. Thus, a semi-structured clinical interview was used to obtain data. The data allows making well-founded assumptions about individual thinking processes and activated mental structures.

The written survey part of Peter and Mike took about ten minutes; the interviews took each about 20 minutes. The whole session was videotaped. The same procedure was undertaken with the students working on two other tasks concerning the concept of basis in other domains and representations, but this is not taken into account in this paper. Peter and Mike are chosen from the whole sample because their solving processes of only one task offer to show different kinds of connections in the mental structures, and because their ways of solving offer a comparison.

A mathematical task and possible approaches

A task given to the students is shown in figure 2.

$$\mathrm{U} = \left\{ \begin{pmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{pmatrix} \epsilon\, \mathbb{R}^3 \mid 2\mathrm{x} = \mathrm{z} \right\} \subset \mathbb{R}^3 \text{ is given.}$$

a) Declare a basis of U.
b) Describe in detail: How did you go on?

Figure 2: Mathematical task given to the students

In the following, approaches of ways of solving the task are briefly pointed out. An approach to solve the task is to choose a vector in U and complete it to a basis by focusing on linear independence. Another approach can focus on linear combination by separating $(\mathrm{x}, \mathrm{y}, 2\mathrm{x})$ according to variables. A starting point could also be to recognise the dimension of U and figure out the number of basis vectors. This could be done by noticing that the equation $2\mathrm{x} = \mathrm{z}$ geometrically describes a plane in $\mathbb{R}^3$ or by noticing that the condition $2\mathrm{x} = \mathrm{z}$ reduced $\mathbb{R}^3$ to dimension two. These are just examples od ideas; of course combinations of the ideas provide adequate approaches as well.

The task is particularly convenient for the reconstruction of mental structures because it complies with the following aspects: The task challenges the students to apply their knowledge conceptually, but also just deals with familiar aspects. The notation of the vector space U is typical and was used in lectures, tutorials, and in the workshop. The task offers various ways of solving. Several connections among ideas of concept can be applied. This is why every student should be able to get access to work on the task at all. There is not just one special idea that needs to be remembered. Moreover, a solution is not available by applying memorised calculations, but refers to conceptual ideas. To put it in a nutshell, the task is rich and simple at the same time.

DATA ANALYSIS AND RESULTS

In a first step, the students' ideas were analysed by using triads. Long chains of triads arise from the whole analysis. They structure the students' thinking processes. The presented analysis consists of short parts removed from the long chains. These examples show the reconstruction of conceptual aspects in a laudable fashion. In a second step, chains become condensed. This process results in the reconstruction of mental structures, which is presented in a net. This is the reconstruction of students' activated mental structure. The focus is on conceptual understanding of the concept of basis. In the original data, all vectors were written as column vectors. In the following I will use row vectors instead of column vectors because it saves space.

The case of Peter

When Peter starts working on the task, he concentrates on the condition $2x = z$, which is given in U. He uses it when creating (1,1,2) and (2,0,4) as possible basis vectors. Then the following interpretant I5 (see table 1) is mentioned.

Triad (T)	Sign (S)	Interpretant (I)	Immediate Object (O)
5	…	Peter: “Basis vectors need to be linear independent.”	linear independence of a set of basis vectors is an essential condition for a basis
6		Peter: “This is why I choose one here and zero there (points at second components in created vectors).”	effect of 0-component concerning linear independence

Table 1: Example 1 of reconstructed triads from Peter’s thinking process

Concerning I5 the reconstructed immediate object is O5. Peter associates this object with the concept of basis. Peter’s interpretant I5 serves as a new sign (S6), which is a part of the next triad. This sign S6 serves to create a next idea, the interpretant I6. The reconstruction of the immediate object O6 is based on I6. In I6 Peter gives an indication of a part of his concept of linear independence. He applies the concept of linearly independence by using the effect of a vector component that is zero. In the following, Peter assumes (wrongly) the number of basis vectors to be three. He refers to $\in \mathbb{R}^3$ in the task and has the opinion that the dimension of every element in the $\mathbb{R}^3$ is three. Then he adds (1,2,2) as a third vector to his basis. After a few seconds he speaks out I20 in table 2.

T	S	I	O
20	…	Peter: “This one is wrong (eliminates (1,2,2) from the set of basis vectors).”	The set consisting of (1,1,2); (2,0,4) and (1,2,2) is not a basis. (1,2,2) is not an element of Peter’s basis.
21		Peter: “I can span (1,2,2) with the other two vectors. If I multiply (1,1,2) by two, divide (2,0,4) by two and subtract this from this, I get this one.” (points at vectors)	‘Not being linear independent’ is used as being able to generate vectors from each other by combining linearly in the head. $\begin{pmatrix}1\\2\\2\end{pmatrix} = 2 \cdot \begin{pmatrix}1\\1\\2\end{pmatrix} - \begin{pmatrix}2\\0\\4\end{pmatrix} : 2$

Table 2: Example 2 of reconstructed triads from Peter’s thinking process

Peter uses the ‘linear independence’-feature again, but refers to another facet than in I6: The possibility to generate a basis vector from the others by combining them linearly. This brings him to reconsider his preliminary basis because basis vectors are generable from others. He uses the concept of linear combination flexibly to argue (I21). Peter adds another third vector and recognises again that the set of basis is not linear independent. Then he gets the idea that two basis vectors will suffice. He refers to the subsigns $\in \mathbb{R}^3$ in the task again and remembers that there are planes in the $\mathbb{R}^3$. Moreover, only two basis vectors are necessary to span planes (which are vector spaces). His idea about two basis vectors is just an assumption until he uses the ‘spanning set’-feature of a basis. He convinces himself that his basis consists of (1,1,2) and (2,0,4) by combining them linearly and focusing on relations among their components. Figure 2 shows Peter’s activated mental structure when he is solving the task.

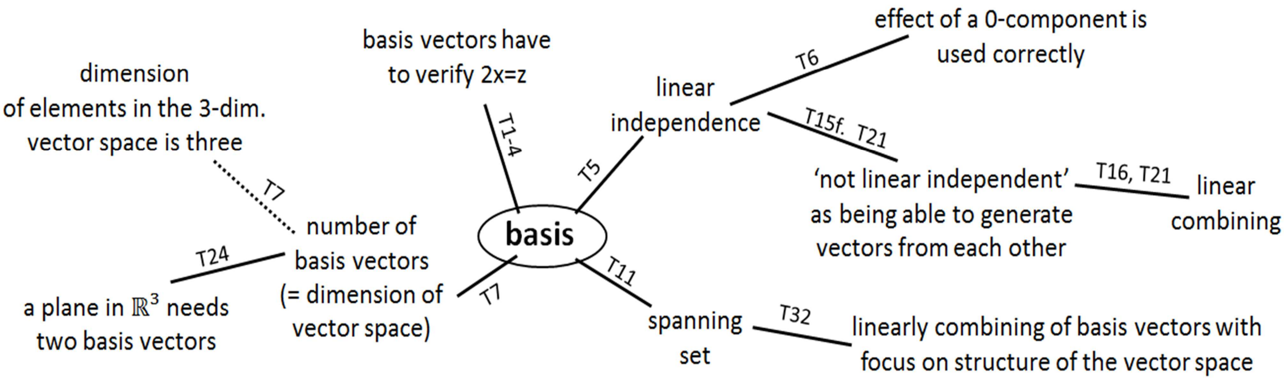

Figure 2: Reconstruction of Peter’s activated mental structure concerning basis

The connection between basis and linear independence is just mentioned in T5. The connections based on linear independence (see figure 2) are rich in content. This enriches the connection between basis and linear independence, too. The connection between ‘number of basis vectors’ and ‘dimension…’ is dashed because the concept of dimension is not used adequately in the context of this task. The inscriptions written at the connection lines declare the number of the triad from which the connection results. Thus, the net offers a look at the order of the problem solving process and the conceptual priorities of Peter. Peter focuses on linear independence.

The case of Mike

Mike mentions that a basis is a linear independent spanning set. Then he focuses on the part of linear independence. He applies a relation between determinant and linear independence which is as follows: $\det(A) \neq 0 \Longrightarrow$ vectors in A are linearly independent. This relation implies A to be a nxn-matrix. In the context of the Mike’s idea A is a 3x3-matrix. He checks if the column vectors of A are basis vectors. Mike applies the relation to four tries of basis sets consisting of (1,0,2), (0,1,0), and a third varying vector. Thereby, he does not analyse any effect of varying his assumed basis vectors. His focus is on the carrying out of the procedure. Using the relation as a very calculus procedure is not productive here. This is why the connection line is dashed in figure 3. Mike cannot conclude a reason why he will never find three basis vectors. He is not successful in finding three basis vectors and has no further access to linear independence. This is why he concludes that the two basis vectors (1,0,2) and (0,1,0) will suffice. The interviewer asks Mike to check if the two vectors build a basis. Then Mike continues with I27 (see table 3).

T	S	I	O
27	…	Mike: The function of basis vectors is „to span U. (…) So, I have to prove if my vectors are a spanning set.”	A basis needs to be a spanning set.
28		Mike writes: span $\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right)$	concretion of idea of spanning set to the context
29		Mike: “I don’t know. This is the span. I don’t know what can be spanned with it. That’s beyond me.”	The representation of the span in I28 is not filled with meaning in this problem solving context.
30		Interviewer: “Linear combination, do you remember this?”	The idea of linear combination is offered.
31		Mike: “I heard about it, but I can’t write anything down.”	Linear combination cannot be used to create any idea.

Table 3: Example 3 of reconstructed triads from Mike’s thinking process

Mike refers to the ‘spanning set’-feature of basis. He applies it to the context of the task. The next interpretant I28 arises from previous thoughts. Mike paraphrases I27 by using the symbolic notation of span. Moreover, he concretises his idea by inserting his own assumed basis vectors into the span. The symbolic notation of span seems to be an empty notation for Mike (see I29 and O29). The interviewer offers the idea of linear combination in I30. I31 shows that Mike cannot use it to create any ideas to go on solving the problem. There is no connection between the symbolic notation of span and linear combination. This is why the relation between span and linear combination is drawn as a ‘disconnection-line’ in figure 3.

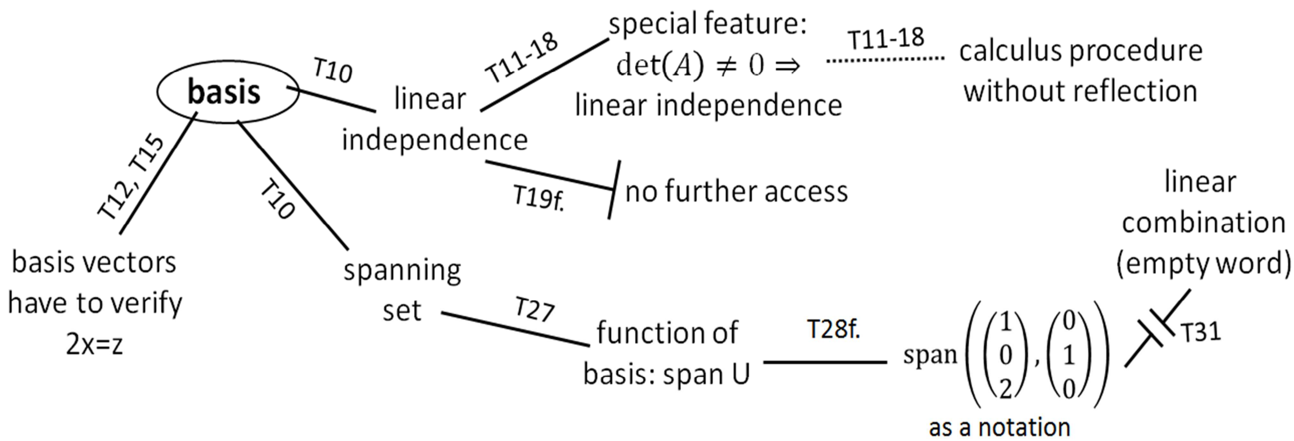

Figure 3: Reconstruction of Mike’s activated mental structure concerning basis

The connection between basis and linear independence is just mentioned in T10. In Mikes further problem solving process (see connections arising from linear independence in figure 3) it is noticeable that this connection is not as substantial as the one in Peter’s reconstructed mental structure. This also applies to Mike’s connection to spanning set. Peter and Mike both focus on linear independence in the solving process and refer to spanning set later. Peter is able to acquire a possible solution. He convinces himself by considering justifications. However, Mike's capacity to act is limited and he cannot justify his set of vectors to be a basis of U.

FINAL REMARKS

In this paper I present my analysing tool. It is based on the semiotic theory of Peirce. The analysing tool allows describing processes of problem solving in a structured and detailed way. The processes are described by chaining triads. The chains serve as a foundation for the reconstruction of students' mental structures that are represented in a net.

Outlook: This procedure is carried out with 15 students and three tasks each. I am interested in understanding the students' mental structure in a more profound way. I plan to identify the ideas of basis the students adapted to their mental structure and establish relationships with the teaching concepts of lectures, tutorials, and the workshop. I will make up types of reconstructed mental structures explicit. By comparing the mental structures, it will be interesting to have a look at the differences between the structures of successful students and the ones who are less successful.

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