STUDENTS' PERSONAL RELATION TO SERIES OF REAL NUMBERS AS A CONSEQUENCE OF TEACHING PRACTICES

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Infinite series of real numbers are a topic that students usually encounter in their first courses of Calculus, however this notion has not received much attention by research, and research has focused mostly on learning difficulties. Our research focuses on the teaching of series and on the consequences of institutional choices on students' learning. After having analysed textbooks and teaching practices, we conjectured the presence of some contract rules in the existing praxeologies to teach series which might have an impact on students' learning. The analyses of the responses to a questionnaire seems to indicate that institutional choices lead the students to learn series without being able to define what a series is, and without being able to identify any application of this notion.

INTRODUCTION

Infinite series of real numbers (*series* in what follows) have been at the heart of the development of Calculus and appear in the programs of the introductory Calculus courses in many countries together with the teaching of sequences, limits, derivatives, and integrals. Series have many applications within mathematics (such as the writing of numbers with infinite decimals, or the calculation of areas by means of rectangles), and also outside of the field of mathematics (as the modelling of situations such as the distribution of pollutants in the atmosphere, or the growing of interests in bank accounts), which may justify their position in Calculus courses.

In Canada, the organisation of education and official curricula is under the jurisdiction of each province. In Québec, compulsory education finishes at the age of 16. For students who wish to pursue university studies, the completion of two years of pre-university studies, called *collégial* is required. For students who want to pursue scientific or technical careers, Calculus is introduced during the *collégial* studies, and it is at this time that series first appear.

Our PhD research (González-Martín, 2006), about the learning of improper integrals, led us to conjecture that students' learning of series could be mostly based on routine aspects. As we discuss in the next section, literature led us to see that most of the scarce research about series has focused on their learning, but not on their teaching. For this reason, we decided to analyse how series are presented in *collégial* textbooks (González-Martín, Nardi & Biza, 2011), and to study how they are taught by *collégial* teachers (González-Martín, 2010), while adopting an anthropological approach (Chevallard, 1999) and acknowledging the key role that the *collégial* institution and its choices play in the learning of series. After our analyses, our results led us to conjecture the existence of some *contract rules* [1] which might have an impact on students' personal relation with series as a consequence of teaching practices. The purpose of this paper is to discuss our results regarding some of these *contract rules* and their impact on students' learning about series.

In the next section, we summarise some previous results about the learning of series found in literature, and we add our own results about their teaching and we discuss the existence of two *contract rules*. We continue by presenting the main points of the anthropological approach that we adopted, in order to discuss later our methodology. After this, we show our data analysis concerning the effect of the *contract rules* on students' learning. We complete this paper with some final remarks.

BACKGROUND

Research literature in mathematics education focusing on series is scarce. Regarding their teaching, series appear implicitly in Robert's (1982) work, where she states that inadequate conceptions of convergence of sequences found in university students in France could be, in part, due to the exercises used in teaching. More recently, Bagni (2000) identified two levels in the construction of a notion in someone's mind (the *operational* and the *structural levels*), and observed that this distinction is not usually considered in the teaching of series.

Other results focus on the learning of series, such as Kidron's (2002) who identifies some difficulties linked to series themselves (such as the use of the potential infinity, for instance), or the confusion between sequences and series (Kidron, 2002; Mamona, 1990). A more exhaustive summary of literature can be found in González-Martín, Nardi & Biza (2011).

As we are interested in the teaching of series and in some possible consequences linked to their teaching, we undertook a research program on different stages. For the first stage, we analysed a sample of 17 textbooks used in *collégial* studies in Québec over a period of 15 years: from 1993 to 2008 (González-Martín, Nardi & Biza, 2011), paying special attention to the praxeologies (see next section) privileged by textbooks. Our main results can be summarised as follows:

- R1. Series are usually introduced through praxeologies which do not lead to a questioning about their applications or *raison d'être*. They do not seem to solve any specific problem.
- R2. Praxeologies tend to introduce series as a tool in order to later introduce functional series, but the importance of series *per se* is usually absent.
- R3. Praxeological organisations tend to ignore some of the main difficulties in learning series identified by research.
- R4. The vast majority of tasks concerning series are related to the application of convergence criteria, or to the application of algorithmic (or previously exemplified) procedures.

The second stage of the research consisted in analysing the use of textbooks that *collégial* teachers make, and whether through their practices they attempt to do something different from what is usually presented in the textbooks (González-Martín, 2010). This is, we tried to analyse whether there are important differences between the *knowledge to be taught* and the *knowledge actually taught*. Our interviews with five teachers revealed that they considered their textbook as adequate for the teaching of series, and that their practices tended to mostly reproduce what was presented in their textbooks. We could even identify some *gaps* between their *didactic intentions* and their practices: for instance, some teachers said that during the teaching and learning of series it is important to feel that the arithmetic of infinity is different; however, in the tasks they privileged, it was not possible to see how these tasks could help their students to achieve this.

As a consequence of the results of these two stages, we conjectured the existence of some implicit *contract rules* in the teaching of series in the *collégial* institutions in Québec. For the purposes of this paper, we will only discuss the two following ones:

> *Rule 1*: "To solve the questions about series that are given, their definition is not necessary".
>
> *Rule 2*: "Applications of series, inside or outside of mathematics, are not important".

These two rules have been chosen for this paper because they are related to two main activities in mathematics: defining [2] and modelling. *Rule 1* could be a consequence of *R2* and *R4*; as series seem to be presented as a tool to introduce other notions, and as the tasks seem to be organised around the application of criteria, students might develop the idea that they do not need to be able to define series in any way, since this knowledge is not required to succeed in the tasks which are proposed. *Rule 2* could be a consequence of *R1* and *R4*; the praxeologies tend to introduce series as a notion which does not solve any particular problem and the focus is established on the application of convergence criteria, without giving any importance to the utility that knowing that a series converges or diverges could have. We believe that these contract rules participate in the characterisation of the institutional relation of the *collégial* institution to series, which might have consequences for students' personal relation to series and for the learning of other notions. We do not advocate that being able to define series in some way, or knowing applications of series, are indicators of a *good* learning of series. Our intention is to better understand the personal relation of the *collégial* students to series as a consequence of institutional practices.

To verify whether these rules have an impact on *collégial* students' personal relation to series, we decided to create a sample of students and to apply a questionnaire (for other examples of this type of work, see for instance Kouidri, 2009).

THEORETICAL FRAMEWORK

As we have said before, our research follows an anthropological approach (Chevallard, 1999), as we recognise the important role of institutional choices in the learning of mathematics, and the repercussions of these choices.

Chevallard's (1999) anthropological theory attempts to achieve a better understanding of the choices made by an institution in order to organise the teaching of mathematical notions. This theory recognises that mathematical objects are not absolute objects, but entities which arise from the practices of given institutions and that every human activity consists in completing a certain type of task. These practices can be described in terms of tasks, techniques used to complete the tasks, technologies which both justify and explain the techniques, and theories which include the given discourses. According to this theory, every human activity generates an organisation of tasks, techniques, technologies and theories which Chevallard designates as *praxeology*, or *praxeologic organisation*. A praxeological analysis allows us to characterise the institutional relation to mathematical notions within given institutions. This institutional relation is mainly forged through the exercises (or tasks), and not only through the theoretical explanations (Kouidri, 2009). Praxeological analyses are useful to describe praxeological organisations, but also to identify the existence of (sometimes implicit) *contract rules*, which are rules that the institution fosters through its practices around a mathematical notion and which contribute to determine the institutional relation to a mathematical notion. This institutional relation and its *contract rules* play an important role in the development of the learners' personal relation to the mathematical notions s/he learns within the institution.

In our case, our praxeological analysis of the teaching of series (both in textbooks and in teaching practices) led us to identify some praxeological organisations in the teaching of series (see González-Martín, Nardi & Biza, 2011), and to identify some implicit *contract rules* which may have a direct impact on the development of the students' personal relation with series.

METHODOLOGY

To verify the impact of the *contract rules* 1 and 2, among others, on the students' personal relation with series, we created a sample of 32 students in their second year of *collégial* studies (where series are introduced) after the teaching of series had occurred. These 32 students come from three different teachers, who we name teachers *A*, *B* and *C*. Our sample consists of 4 students from teacher *A* (referred to as students A1 to A4), 14 students from teacher *B* (referred to as students B1 to B14), and 14 students from teacher *C* (referred to as students C1 to C14).

We constructed a questionnaire with 10 questions, aiming to assess the students' learning about series, as well as to verify our conjectures about the impact of the *contract rules* on their learning. The questionnaires were administrated in May 2011 during one of their courses (approximately 55 minutes in duration), and the students participated voluntarily.

In this paper, we discuss the students' responses to the three following questions:

Question 1:

Define the notion of "numerical series" or "infinite sum".

Question 3:

Name at least two different applications of series in the field of mathematics.

Question 4:

Name at least two different applications of series in a field different than mathematics.

For one of these applications, create a realistic problem whose resolution requires the use of series. Solve the problem.

In the next section, we present and comment on the results obtained to these three questions.

DATA ANALYSIS

In this section, we present and discuss our data for each of the three questions.

Question 1

The distribution of responses to this question is the following:

Correct definitions with her/his own words explicitly mentioning a sum or the addition of terms.	B4*, B13
"Let $\{a_n\}$ be a sequence, we write: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$"	A1, A4 B1
"An infinite series has the form: $\sum_{k=1}^{+\infty} aK^{k-1}$. We multiply a constant by a number till infinity. We add all the terms."	C11
Other definitions mentioning the necessity of defining an equation, or finding some logic among the terms. Some of these definitions illustrate confusion between sequences, series and other notions.	A2 B3, B12, B14 C3*, C4, C5, C8, C9, C12, C13, C14
Definitions showing some confusion between sequences and series.	B2, B6, B9 C1, C6*

Incorrect definitions	A3 B5, B7, B8, B10, B11 C2, C7, C10*

Table 1: Responses to Question 1

Only five students (A1, A4, B1, B4, B13) provide definitions with no erroneous elements. However, mentioning that the sum could converge or diverge does not seem to be an important thing to mention for the students, as only the students marked with * mention this fact. Other students (particularly those from teacher *C*) seem to associate series with the existence of a formula or regularity, which might be a consequence of the praxeologies used. For many students, in their discourse, there seems to be some confusion between sequences and series, or in the use of some vocabulary: “it’s the sum of sequences” (B2), “infinite series, it’s series in which we add the terms indefinitely” (B6), “numerical series: a sequence of numbers having some logic. Infinite sum: when you add numbers infinitely” (C4)...

We also wish to highlight the fact that most of the students provided informal definitions in their own words, and only A1, A4 and B1 seemed to be able to provide a correct, symbolic definition, although none of them mentioned the possibility of the sum being finite or finite.

These responses seem to indicate that *Rule 1* has an important effect on many students’ learning, shaping their personal relation to series. The praxeological organisation necessary to solve many tasks is usually structured around the application of known criteria (*R4*), therefore the students are not required to remember or to understand what a series is to apply these criteria.

Question 3

The distribution of responses to this question is the following:

No response or a rhetorical response (“good question”)	B10 C4, C9
I don’t know	A1, A2, A3, A4 B1, B8, B11 C1
“It’s too abstract. I just apply rules”	B11, B12
Helps to find some values (like e, π, $\ln x$, $\sin x$)	B4, B9, B13
Mention of some convergence criteria as applications	B5, B7
Helps to make approximations (to curves, in a calculator, or through Taylor polynomials)	B6, B12 C14

Helps to calculate areas	B8 C3, C10
Other	B2, B3, B14 C2, C5, C6, C7, C8, C11, C12, C13

Table 2: Responses to Question 3

Our analysis of textbooks led us to state that textbooks seem to teach students to determine the convergence or the divergence of some series, but that this task has no utility or purpose (*R1*, *R2*) (González-Martín, Nardi & Biza, 2011). The responses to this question seem to confirm our initial impressions as most students do not seem able to clearly state any mathematical applications of series. The students who are able to give some applications (like finding some values, or making approximations), do not seem to be able to give details about how series are used (so, probably, they just heard their teacher quickly mention some applications).

One application which appears in certain textbooks (modelling the distribution of medication in the blood) is vaguely remembered by just one student: “I am not sure, but we can use them to determine or to know the amount of medication present in the organism of a person” (B14).

We believe it is also very significant that the four students of teacher *A* acknowledge not knowing any application. We can also see that almost all the students who mention the use of series to calculate some values, or to make approximations, are students from teacher *B*. In sum, the students’ responses lead us to believe that possibly, teacher *A* did not mention any application during his teaching while teacher *B* probably mentioned some applications without giving details, and finally, that teacher *C* made some links with integrals and Taylor polynomials.

The possible consequence of *Rule 2* is that students are unaware of the applications of series. Data seem to indicate that this consequence has occurred among the students in our sample, and the responses to question 4 seem to confirm this conjecture.

Question 4

The distribution of responses to this question is the following:

No response	B6, B10, B12 C7, C9, C10
I don’t know	A1, A2, A3, A4 B4, B9, B11, B13 C1, C4

Example of the ball which bounces indefinitely	C2, C8
Example of the ball which bounces indefinitely (developing some calculations)	C6, C11, C14
In Chemistry: “every second, half the number of molecules having reacted the second before, react...”	C3
“The digestion of a substance is made in a specific way. After one hour, half the substance is digested. Two hours later, only a quarter of the substance remains...”	B3
Pharmacy: “A medication which is taken every day. Calculate the amount of the medication in the body of a person in this context”.	B1
Besides mathematical applications, I don’t know	B7 C5
Other	B2, B8, B14 C6, C12, C13

Table 3: Responses to Question 4

None of the students were able to create a realistic problem, to model it and to solve it using a series. As in the previous question, we observe some regularities in the responses: none of the students from teacher *A* can give a response; all of the students who mention the ball with infinite bounces are from teacher *C*; again, the example of the distribution of medication comes from a student from teacher *B*.

Just as the textbooks analysis and the teaching practices analysis suggested, applications of series to model some situations seem to be generally absent from praxeological organisations, which seems to strengthen the development of *Rule 2* and its impact on students’ learning. Even if some students seem to be aware (in a vague way) of some applications (inside or outside of the field of mathematics), they do not seem to be able to further develop them or to give specific details, probably because tasks requiring them to do so are absent from the praxeologies privileged in the *collégial* institution.

FINAL REMARKS

In spite of the importance of series for the development of modern mathematics, the teaching of series in the *collégial* institution seems to reduce them to a set of criteria and algorithms used to solve tasks which do not seem to have any purpose (“once we know this series converges, what do we do with that knowledge?”).

The analysis of the praxeological organisation of the teaching of series both in the textbooks and in the teaching practices at the *collégial* level led us to conjecture the existence of some implicit *contract rules* having an impact on students' learning. Our results seem to support our conjecture, meaning that students learn to solve some questions concerning the convergence or the divergence of given series, without:

- Clearly being able to state what a series is.
- Clearly knowing what utility solving this task could have.

Even if the definition of series appears in the textbooks (and usually this is the first encounter of students with series), and some textbooks present certain applications (in a very vague way), praxeological organisations do not seem to clearly place any importance on these elements. However, students are later required to construct other mathematical notions (like power series and Taylor polynomials) from a notion they are barely able to define, and one key mathematical activity (modelling) is completely absent from these practices.

Our paper discusses the responses to just three questions from our questionnaire. Further questions aimed at verifying other conjectures, such as:

- Students are generally able to solve routine tasks which require the application of some convergence criteria, but how do they interpret them, especially in the cases when we cannot conclude anything?
- The notion of convergence, especially in other settings which are not symbolic (e.g. geometric setting), is not clear for students regarding series.
- The identification and the manipulation of series in settings other than the symbolic setting are not evident for students.

We expect to continue our analyses and to provide a more detailed portrait of the learning achieved by students within the *collégial* institution, as well as to identify other *contract rules* that praxeologies might be developing and their impact on students' learning.

An awareness of the consequences of practices being privileged at the *collégial* level could be helpful in order to begin a discussion about how series are taught and whether the students' difficulties in learning series are taken into account in teaching practices and finally, the possible consequences of current practices. We hope that our results can lead to the development of more research focusing on the teaching and learning of series, as well as to stimulate a discussion among members of the teaching community at the post-secondary level about the practices which are privileged and their consequences in the learning achieved by the students.

NOTES

1. "*Règles de contrat*" in the French literature.

2. Some authors have argued the importance of definitions to learn mathematical notions. See, for instance: Harel, G., Selden, A. & Selden, J. (2006). Advanced Mathematical Thinking. Some PME

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ACKNOWLEDGEMENTS

The research reported in this paper has been funded by the program Nouveau chercheur du Fonds Québécois de Recherche sur la Société et la Culture (FQRSC) – ref. 128796.

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