INTERACTIVE RECONSTRUCTION OF A DEFINITION

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For future mathematicians to be able to construct proofs, they must understand what a correct mathematical definition is and how to use it properly. In this sense, a task was proposed to college students for reconstructing the definition of even numbers through discussion in small groups and then a socialization of the answers guided by the teacher was carried out. Both interactions were recorded and analyzed entirely with the RBC-C model (Schwarz, Dreyfus & Hershkowitz, 2009) for documenting how the reconstruction process of a definition takes place. The results show that the students had difficulties to manage mathematical language correctly and that the definitions they produce are erroneous; they needed the teacher's help to propose constructive definitions.

INTRODUCTION

One of the most important aspects in the process of doing proofs is language accuracy, which means realizing how arguments are presented and how concepts are defined (Vinner, 1991; Mariotti & Fischbein, 1997). This requires students to have a clear idea of what a mathematical definition is. Pinto & Tall (1999) indicate that students handle formal definitions in two different ways: by giving them meaning through consideration of examples or by extracting their meaning through manipulation and reflection.

Regarding ways of defining, De Villiers (1998) classified definitions considering ones that define *descriptively* (a posteriori), if "the concept and its properties have already been known for some time and is defined only afterwards… A posteriori defining is usually accomplished by selecting a subset of the total set of properties of the concept from which all the other properties can be deduced" (De Villiers, 1998, p. 250). The other way is to *define constructively* (a priori). This takes place when a "given definition of a concept is changed through the exclusion, generalization, specialization, replacement or addition of properties to the definition, so that a new concept is constructed in the process" (De Villiers, 1998, p. 250).

These forms of defining are distinguished by their function. Whereas the function of a posteriori defining is to systematize existing knowledge, the main function of a priori defining is the production of new knowledge.

When students define they must generalize some facts they appeal to two types of generalizations: result pattern generalization (RPG) and process pattern generalizations (PPG) (Harel, 2001). The first one is a way of thinking in which one's conviction is based on the regularity found in processes whereas the other is based on the regularity in the results, obtained, for example, by substitution of numbers in a formula.

CONCEPTUAL FRAME

Schwarz, Dreyfus & Hershkowitz (2009) defined Abstraction in Context (AiC) as "a vertical activity for the reorganization of previous mathematical constructs within mathematics and by mathematical means so as to lead to a construct that is new to the learner"(p. 24).

A process of abstraction has three stages: the need for a new construct, the emergence of a new construct and its consolidation. The need may arise from an intrinsic motivation to overcome obstacles and contradictions, surprises, or uncertainty. In the second stage the new construct emerges by means of three observable epistemic actions: Recognizing (R-actions), which recognizes that a specific prior construct is relevant to the problem or situation at hand; Building-with (B-actions), if it acts on or with the constructs recognized for achieving the goal of understanding a situation or solving a problem; Constructing (C-actions), to fit and integrate previous constructs by vertical mathematization to produce a new construct. Constructing refers to the first time the new construct is used or mentioned by the learner. In this process, Recognizing is nested within Building-with actions, and these in turn in the Constructing actions that can be nested in other Constructing actions of higher level. Finally, the third stage, corresponding to Consolidation, is a long-term process that occurs when the construct is mentioned, constructed or used after a constructing action is observed. This last stage is characterized by personal evidence: self-confidence, immediacy, flexibility and care when working with the construct (Dreyfus & Tsamir, 2004) and also when the language is increasingly more accurate (Hershkowitz, Schwarz & Dreyfus, 2001), although Kidron (2008) and Gilboa, Dreyfus & Kidron (2011) consider that the increase in language precision is characteristic of the construction stage itself and not just the consolidation stage.

In AiC these epistemic actions are known as the RBC model (Recognizing, Building with, Constructing) and the RBC-C model with the second C corresponding to the consolidation phase.

Below we present an activity about the process of reconstructing the definition of simple mathematical concepts as a prelude to using these concepts in a proof. Our objective was to describe how this process is developed in order to reach a minimal mathematical definition (Zaslavsky & Shir, 2005). Although the mathematical concepts proposed to the students were elementary, the teacher observed that previous work with definitions is needed, in view of the difficulties faced by the students with the use of mathematical language when they are proving.

We analyzed the production of various small-groups regarding different definitions. Specifically, in this paper we present the productions of one of these groups concerning the concept of even numbers analyzed according to the AiC model

(Hershkowitz, Schwarz & Dreyfus, 2001). To get an overall picture of the entire process followed, we also present a whole class interaction leaded by the teacher.

Thus, the question this study aims to answer is: What are the epistemic actions that arise in the course of small group and whole class interactions or during the process to reconstruct a minimal definition of even numbers?

METHODOLOGY

The activity described below is part of a broader research about the introduction of mathematical proofs to university undergraduate students of mathematics. In this context sets of activities have been designed and experimented in a university classroom during eleven sessions. The activities were designed by the researchers and put into practice in the teacher's classroom. The activity in this paper was developed in the first session, in order to know how definitions are handled and generated.

The construction of proofs requires an understanding of the concepts involved and the appropriate use of their definitions. Different researchers have found that students do not understand the content of relevant definitions and how to use them for writing proofs (Moore, 1994). In an exploratory study (Alvarado & González, 2010) it was found that students do not use the definitions of mathematical objects to prove a statement in which they are involved; rather, they use an example of the object, a formula that represents it or some characteristic of the object. Edwards and Ward (2004) suggest that the special nature of definitions should be treated in introductory courses about proofs as a content in its own right and suggest that its effectiveness must be investigated in future research.

Specifically, in this activity, students had to reconstruct definitions for already known mathematical concepts or objects through negotiation in small groups and then they had to share their definition in a whole class discussion mediated by the teacher. The construction process carried out in each of the teams allowed the students to verbalize their ideas, thus making it possible for us to analyze the evolution of this process in the light of the RBC-C framework. This experiment involved 23 students that were studying the first semester of Applied Mathematics in the University of Juarez in Mexico. The group was divided into 9 small-groups of 2 or 3 students.

DATA ANALYSIS

Below we describe, characterize and analyze the interactions that took place in one of those small-groups (team X) as they tried to reconstruct the definition of even numbers followed by whole group socialization influencing their small group work.

Small-group interactions

In the dialogues in this section only participate students. The contributions of each of them were consecutively numbered to refer to them easily in the text. At the beginning of the interaction, the students suggest a rather poor definition [1]. They evoke the concept of division (R-action) and relate the concept of even numbers with composite numbers using some examples to reconstruct the definition of even numbers to show their truth.

[1] Then an even number is one that can be divided by itself.

[2] And others.

[3] And others?

[4] Yes, because for example 10 can be divided by itself, by 5, by 2.

[5] ... the bigger the number the more numbers it can be divided by.

[6] Look, another even number.

[7] 18.

[8] 18, by 18, by 9, by 2.

[9] What else?

The generalization process being used is based on patterns of results (RPG) and not on processes (PPG) (Harel, 2001). The above examples led them to guess a more appropriate definition [10] which is verified again (B-action) with other numbers.

[10] All even numbers can be divided by 2, right?

[11] Yes, I think so.

[12] By 2 yes, 4 yes, 6 also, 8, 14 also.

By linking (B-action) the above [1 and 10] with the fact that 2 is a prime number [17, 18 and 20] they produce a "definition" [14, 19] in which similarities with the definition of prime number are seen: both of them have two divisors [19 and 20]. This is caused by their difficulties with the use of quantifiers.

[13] So even number. [he writes]

[14] Is that which is divided, which can be divided by ... oh no!

[15] By itself.

[16] And another number.

[17] The number 2 we did not say.

[18] By 2, right?

[19] By itself and by 2.

[20] By 2, isn't it? It's a prime number.

The definition produced [21] is obtained by a B-action to verify similarities between the examples found earlier.

[21] Yes that's right [pointing to the examples above], all numbers can be divided by themselves and by 2.

Another student suggests another definition [22] which is considered a C-action (C_{X0} as this is the first C-action shown by group X), but their peers did not take it into account.

[22] Even numbers are those who go two by two. Because 4, 6, 8 go two by two.

Similarly, we include the wrong definition they proposed of odd numbers, which is accepted [23] without discussion by the other peers. It can be seen the similarity with the definition of even numbers and the confusion with prime numbers.

[23] You can only divide it by itself and by 1.

Whole-group interactions

Small-group work allows for an interaction that simultaneously provides advances and limitations. Some of these limitations will be overcome during the whole-group discussion because it addresses aspects or views not suggested during the small-group interactions, such as an input from a student of small-group X which was not taken into account, and will now be taken up by the teacher. During the socialization the teacher plays an important role supporting the development of mathematically productive discourse and progress in the learning processes.

To organize the discussion, the socialization starts by presenting the definition contributed by small-group Y. Then the definition of X is discussed, and extending it comes into play with the production of small-group T. Finally, we discuss the productions of small-groups V, W and Z which allows the construction of another definition.

This began with a spontaneous and intuitive notion: a list of numbers from zero that is increased by 2. This C-action C_0 [24] built by team Y is equivalent to a C-action that occurred during the interaction in small-group X (C_{X0} [22]). Demanding an explanation of the meaning with the purpose of clarifying the language used [25] constitutes an R-action. Students will refine the language using different words and examples that serve as B-actions giving rise to a C-action [31] C_1: a definition of even numbers that for the time being is neither accepted nor rejected pending further "definitions".

[24] Student: They are the numbers that ascend from zero 2 by 2. [small-group Y]

[25] Teacher: "Ascend", what does this mean?

[26] Student: That they go up.

[27] Teacher: Okay, explain.

[28] Student: That we are adding and adding.

[29] Teacher: For example.

[30] Student: 2, 4, 6...

[31] Teacher: Okay. So they are the numbers to which you keep adding 2 beginning with zero. Any other input?

The teacher shows that this definition can be completed and formalized. It should be noted that while the original idea of the students is respected, the teacher uses the role of authority to extend the mathematical knowledge in a *constructive way* of defining (De Villiers, 1998).

[32] Teacher: It turns out that we only get positive even numbers, so to complete the definition with the negative even numbers, I have to subtract to the left.

[33] Teacher: But that thing, of using "ascend" – it’s a very ambiguous word. Let's get used to the fact that in mathematics we must be very precise.

[34] Teacher: In this case we can define it as the sequence of integers where starting from zero we add and subtract two units successively. This would look like this [he writes on the blackboard] {....-8, -6, -4, -3, -2, 0, 2, 4, 6, 8... }.

[35] Teacher: It can also be formalized as $\{0 \pm 2n \,/\, n \in \mathbb{Z}\}$. This means that *n* is an integer or that *n* varies within the integers.

An R-action takes place when it is recognized on the one hand that negative numbers must be included, and on the other hand, that appropriate language must be used. This is concluded with two C-actions, C_2 [34] and C_3 [35]; the second as an extension of the first. In this case the teacher does not encourage the interaction with students to build the definition.

Returning to the definition produced by X, the teacher tries to adjust it to produce a more economical definition. To do so, the students should realize that it is not necessary to include the number 1 in the definition.

[36] Student: They are the numbers that are divisible by themselves and by 2. [small-group X]

[37] Student: And by 1.

[38] Student: And 1.

[39] Teacher: Let’s go over that again. Do we need to say that they "are divisible" by themselves and by 1? In fact, any number can be divided by itself and by 1. So this is not an exclusive feature for even numbers. Let’s see, what distinguishes the even numbers?

This last statement and the students’ answers to the teacher's suggestions are an attempt to clarify the language to formulate a more operational definition.

[40] Student: That even numbers can be divided by 2.

[41] Teacher: Right, but then we have a problem, for example, I can divide 3 by 2, can’t I?

[42] Student: Yes.

[43] Teacher: Then the word “divided by” must be refined.

[44] Student: The matter is that we must get an integer.

[45] Student: Yes, a number that has no decimal.

[46] Student: That gets me zero as remainder. [small-group V]

[47] Teacher: ...A number that when divided by 2 the remainder is...

[48] Student: Zero.

A B-action [40] is activated when an exclusive feature of the even numbers is mentioned. During this time [41-43], with R-actions, a negotiation of the meaning of the term "can be divided" is elicited because for some students it means that the quotient must be an integer [44 and 45], which does not guarantee that the remainder is zero, even though a suitable formulation is eventually adopted [46]. This expression appears repeatedly in the interactions of most of the small-group work with the idea of expressing that the remainder of a division must be zero. This can be attributed to the fact that when 3 is divided by 2, for example, it is common to hear young children say "can’t be done" or "it won’t fit", and thus the spontaneous use of language is maintained until university level.

Moreover, although the contributions of X and Y teams are discussed, the teacher takes advantage of this discussion to include another definition produced in team T, when this happens a new C-action C_4 [47 and 48] takes place.

Finally we discuss what small-groups V, W and Z did to build another definition refining the language.

[49] Teacher: Let's see, what happens if I divide 10 by 2, what will it be? How much is left over?

[50] Student: 5 and zero as remainder.

[51] Teacher: So I say that it is an even number. Remember when you were in elementary school you were told: "check your calculations to see if they are correct". What operation did you have to do?

[52] Student: The inverse operation of division is multiplication, adding what is left over.

[53] Teacher: For example here, in this case $2 \times 5 + 0$, and since the remainder is zero, we just write 2×5 and say that 10 looks like 2×5. So, what form do the even numbers take?

[54] Student: 2 times "something".

[55] Teacher: Does 12 take the form 2 times "something"?

[56] Student: Yes 2 by 6, so it is even.

[57] Teacher: Does 13 take the form 2 times "something"?

[58] Student: No.

[59] Teacher: So, even numbers, what do they look like?

[60] Student: 2 times an integer.

[61] Teacher: Well, then we have another definition, even numbers are those that take the form $2\times n$ where n is an integer [writing on the blackboard] Even numbers look like $\{2n / n\in Z\}$. The $n\in Z$, this is interpreted as n ranging over the integers or n belonging to the integers. This is a clear way to say it and if this doesn't happen, what occurs?

[62] Student: Well, is not an even number.

Although previously were obtained definitions of even numbers accepted by the group, now the idea is to achieve a correct definition obtained constructively. This is how the C-action C_5 [61] takes place, built on B-actions oriented and provoked by the teacher presented as a string of examples in which transitions are made from a particular example [49], to a process [51], to generic constructions [59, 60] via a check that ensures that non-examples are identified and ruled out [57, 58].

The following diagram summarizes the progression between the different definitions produced by the teams to reach more operational definitions.

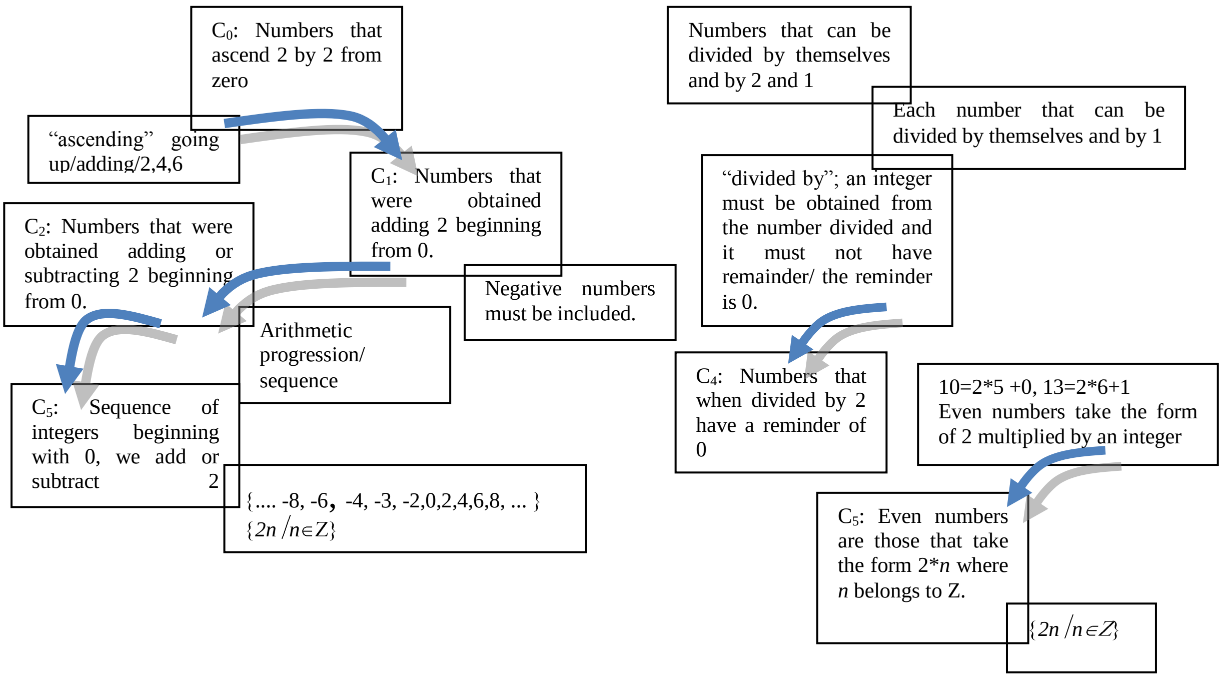

Diagram 1. The process of generalizing the definition of even numbers in whole-class.

The confusion between different types of numbers and the difficulties managing their definitions is present when solving other activities. For instance, when these students have to prove that "For any whole number n, n^2+n+41 is a prime number" they relate prime numbers with odd numbers saying things like “3 times 3 equals 9, and 3 are 12, plus 41 are 53, 53 is an odd number so is prime”.

CONCLUDING REMARKS

The RBC model is a tool for analyzing the process of reconstructing a definition from a more informal view of mathematics to make definitions that require more formal reasoning (Gravemeijer, 1999). Even in the simplest situations, students use examples as verifiers and conjectures derived from the results rather than processes (Harel, 2001). This is shown essentially in the B-actions. Vinner (1991) states that the development process of a concept involves two cognitive mechanisms: identifying similarities and distinguishing differences. However, students look only at the former and disregard the latter completely. Some difficulties with quantifiers are perceived, resulting in the formulation of inconsistent definitions in the small-group interaction.

When the teacher leaded whole-class discussion it is possible to produce a definition focused on the generalization of patterns based on the regularity (PPG) and not on the RPG obtained from each example (Harel, 2001). In this sense Harel (2008) point out that only the students that generalize across PPG suggest the required arguments for the construction of proofs by mathematical induction. In this same sense, the definitions proposed by students in small-groups were mainly obtained descriptively whereas during the whole-class socialization, where the teacher guides the discussion, sometimes the definitions are obtained descriptively and other times in a constructive manner (De Villiers, 1998).

Finally, we argue that negotiated defining in small-groups helps the students to extend their understanding about the meanings of the terms. On the other hand, this activity suggests an issue that should be taught and learned in school classrooms in order to improve the mathematical communication, instead of assume that students already have those skills.

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