UNDERSTANDING UNDERSTANDING EQUIVALENCE OF MATRICES

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The title of the paper paraphrases the title of the famous paper by Edwina Michener "Understanding understanding mathematics". In our paper, we discuss what it means to "understand" the concept of equivalence relations between matrices. We focus on the concept of equivalence relations as it is a fundamental mathematical concept that can serve as a useful framework for teaching a linear algebra course. We suggest a definition of understanding the concept of equivalence relations, illustrate its operational nature and discuss how the definition can serve as a framework for teaching a linear algebra course.

INTRODUCTION

One of the major dilemmas that teachers of linear algebra face is whether to start with abstract concepts like vector space and then give concrete examples, or start with concrete applications like solving systems of linear equations and then generalize and teach more abstract concepts. Our personal preference is to start with systems of linear equations since they are relatively easy-to-understand and are connected to the students' high school experience. Unfortunately, for some students this only delays the difficulty of abstraction (Hazzan, 1999). The students also often tend to consider less and more abstract topics of the course as disjoint ones. Since the concept of equivalence relations appears both in concrete and abstract linear algebra topics, we think of equivalence relations as an overarching notion that can be helpful in overcoming these difficulties.

In this paper we first suggest a review of topics, in which the notion of equivalence relations appear in high school and in a university linear algebra course and then theoretically analyse what it means to understand this notion, in connection with the other linear algebra notions. For this purpose we suggest a definition of understanding the concept of equivalence relations in linear algebra and argue, by means of presenting mathematical tasks aimed at testing particular aspects of the definition, that it can be operationalized. The paper is concluded with suggestions for future empirical research.

EQUIVALENCE RELATIONS

Examples of equivalence relations known, latently, to high school students include equality of numbers and algebraic expressions, and congruence and similarity of geometric shapes. Enriched mathematics high school curriculum may also include congruence modulo in elementary number theory and equivalence of (systems of) equations.

Equivalence relations between matrices are ubiquitous. Equivalence of systems of linear equations is usually the first time when a university linear algebra student explicitly encounters the concept (e.g., Berman & Kon, 2000; Carlson, Johnson, Lay & Porter, 1993; Hoffman & Kunze, 1972). The concept of row equivalence of matrices is introduced in this connection.

The concept of column equivalence of matrices is introduced for the sake of symmetry, and an experienced lecturer would emphasize that elementary row operations transform a system of equations to an equivalent one, whereas elementary column operations do not. Matrix equivalence naturally appears in connection with rank, matrix similarity – in connection with eigenvalues, and matrix congruence – in connection with quadratic forms. Figure 1 describes the logical-hierarchical connections between these types of equivalence relations. An arrow in the figure between a relation α and a relation β means that if relation α exists between two matrices, then so does relation β; the matrices P and Q are invertible.

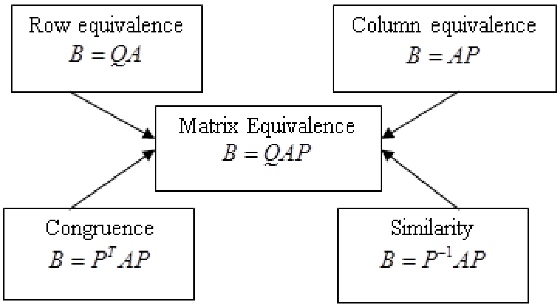

Figure 1: Examples of matrix equivalence

Other types of equivalence relations between matrices are restricted to complex matrices. These types are presented in Figure 2.

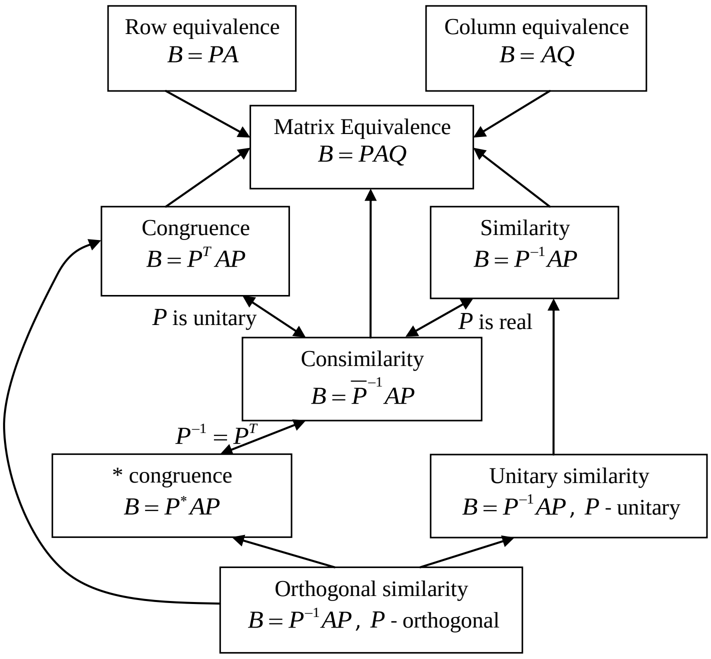

Figure 2: Equivalence relations of complex matrices

As in Figure 1, the matrices P and Q are invertible, and a one-side arrow between a relation α and a relation β means that if relation α exists between two matrices, then so does relation β. A two-side arrow in Figure 2 between a relation α and a relation β, accompanied by condition γ, means that under condition γ the relations are the same. By an orthogonal matrix we mean a real matrix P satisfying $P^{-1} = P^T$. Note that in the condition $P^{-1} = P^T$ that accompanies the arrow between consimilarity and *congruence, the matrix P is not necessarily real.

UNDERSTANDING EQUIVALENCE RELATIONS

Michener (1978) accounts a mathematician's perspective of what it means to "understand" as follows:

> When a mathematician says he understands a mathematical theory, he possesses much more knowledge than that which concerns the deductive aspects of theorems and proofs. He knows about examples and heuristics and how they are related. He has a sense of what to use and when to use it, and what is worth remembering. He has an intuitive feeling for the subject, how it hangs together, and how it relates to other theories. He knows how not to be swamped by details, but also to reference them when he needs them. (p. 361)

We apply this perspective to conceptualizing students' understanding in linear algebra. As will be evident shortly, our conceptualization of understanding is also stimulated by the work of Skemp (1976, 1986) on *instrumental understanding/relational understanding*.

Generally speaking, we refer to *understanding* a concept in a given mathematical subject as the capability to provide different representations of the concept, link it to other concepts by logical-hierarchical relations, and apply it in central issues of the subject. Specifically, in the context of equivalence relations in linear algebra, we suggest the following definition:

Understanding of an equivalence relation of matrices consists of:

- *Formal understanding* – capability to recall (on demand or when needed in problem solving) its formal definition(s).
- *Instrumental understanding* – capability to transform a matrix to an equivalent one.
- *Representational understanding* – capability to recall (on demand or when needed in problem solving) properties of an equivalence class and capability to find simple representatives of the equivalence classes.
- *Relational understanding* – capability to relate the relation to other concepts, including other equivalence relations (e.g., Figure 1 and Figure 2 above).
- *Applicational understanding* – capability to identify (not necessarily to solve) problems in which the relation may be useful.

Note that there is no hierarchical relation between the different types of understanding. For example, it is possible that a student may possess representational understanding without instrumental one, or vice versa. Also it is clear that each type of understanding has its own spectrum of deepness, from a basic to an advanced one. Deepness of understanding can be characterized in terms of available arsenal of relevant proofs, generic examples and problem-solving strategies. For operational reasons, we consider the following levels of understanding. A basic level of *formal understanding* is when the student can recall the relevant definitions. In case that there are several definitions, the ability to prove that they are equivalent demonstrates an advanced understanding. The levels of *instrumental understanding* can be characterized by the fluency of performing the transformations. A student at a basic level of *representational understanding* can only recall simple representatives. A more advanced student can prove existence and uniqueness of the representatives. A basic level of *relational understanding* presumes that a student knows how the equivalence relation relates to other concepts. A more advanced level is expressed by the ability to prove these relations. An advanced level of *applicational understanding* is when a student not only knows in which problems the concept may be useful, but also knows how to solve some of the problems.

These definitions can be operationalized since each of the above types and levels of understanding can be evaluated by means of appropriate tasks. In the next section we give some examples.

UNPACKING THE DEFINITION BY MEANS OF TASKS

As examples we explain the five types of understanding for row equivalence, matrix equivalence and matrix similarity. In addition, for matrix equivalence, we show how formal, representational and applicational understanding can be evaluated by tasks. For matrix similarity we discuss the pedagogical consequences of the lecturer's decision to stress some types of understanding more than the others. We also discuss the potential of teaching orthogonal similarity for creating an overall picture of the course through promoting its relational understanding.

Row equivalence

Formal understanding: the students are capable of recalling that a matrix B is row equivalent to a matrix A if B can be obtained from A by a finite number of elementary row operations. They should also know that this is the same as $B = QA$, where Q is invertible.

Instrumental understanding: the students can perform elementary row operations and know how to transform a given matrix to a row equivalent one.

Representational understanding: the students know that every matrix can be reduced to a row echelon form and is row equivalent to a unique matrix in a row reduced echelon form, and thus A and B are in the same equivalence class if and only if they have the same row reduced echelon form.

Relational understanding: the students are capable of associating row equivalence with systems of linear equations, can recall that row equivalent matrices have the same rank, but that the converse is not true, and that row equivalence implies matrix equivalence. The students also know that matrices of the same order are row equivalent if and only if they have the same row space.

Applicational understanding: the students can recall that a system of linear equations can be solved by reducing the augmented matrix to a row equivalent row reduced matrix (Gauss elimination) or to its row reduced echelon matrix (the Gauss-Jordan method). More advanced applications include vector independence, finding a basis and matrix inversion.

The concept of row equivalence is a basic concept and thus it is important that all students will develop all the five types of its understanding, at least at the basic level.

Matrix equivalence

The concept of matrix equivalence is less basic and it is not necessary to emphasize it in a very basic linear algebra course. However, in a more advanced course it makes sense to develop some of the following:

Formal understanding: the students recall that matrices A and B are equivalent if one can be obtained from the other by a finite number of elementary, row or column, operations, or, equivalently, if $B = QAP$, where P and Q are invertible.

Instrumental understanding: the students can perform row and column elementary operations.

Representational understanding: the students know that every $m \times n$ matrix of rank r is equivalent to an $m \times n$ matrix of the form $\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$, where I_r is the $r \times r$ identity matrix.

Relational understanding: the students know that $A, B \in F^{m \times n}$ (The $m \times n$ matrices over the field *F*) are equivalent if and only if they have the same rank, that equivalent matrices represent the same linear transformation, and that the row equivalence, column equivalence, similarity and congruence are special cases of matrix equivalence (see Figure 1).

Applicational understanding: the students know that reducing a matrix to its simple representation $\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$ can be useful in problems involving rank.

As an example, consider the following classical proof of the fact that for A, $B \in F^{n \times n}$, AB and BA have the same characteristic polynomial:

Proof: Suppose $rank\, A = r < n$ (If $r = n$, AB and BA are similar and thus have the same characteristic polynomial). Then A is equivalent to $\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$, i.e., there exist invertible matrices P and Q such that $PAQ=\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$. Let $Q^{-1}BP^{-1}=\begin{pmatrix} C & D \\ E & F \end{pmatrix}$, where C is $r\times r$. Then $PABP^{-1}=PAQQ^{-1}BP^{-1}=\begin{pmatrix} C & D \\ O & O \end{pmatrix}$, and $Q^{-1}BAQ=Q^{-1}BP^{-1}PAQ=\begin{pmatrix} C & O \\ E & O \end{pmatrix}$. Hence AB is similar to $\begin{pmatrix} C & D \\ O & O \end{pmatrix}$ and BA is similar to $\begin{pmatrix} C & O \\ E & O \end{pmatrix}$. The matrices $\begin{pmatrix} C & D \\ O & O \end{pmatrix}$ and $\begin{pmatrix} C & O \\ E & O \end{pmatrix}$ have the same characteristic polynomial, and thus so do AB and BA.

In terms of understanding matrix equivalence, knowledge of the above proof requires basic level of formal and representational understandings, and an advanced level of applicational understanding. A lecturer interested in developing the applicational understanding of matrix equivalence may include this proof in the course. A lecturer who needs time for other purposes may use other proofs. We remark that the proof can be used for evaluating different types of understanding when divided into sub questions. For example, it can be represented as follows:

Question A: Prove that an $n\times n$ matrix A of rank r is equivalent to $\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$, where I_r is the $r\times r$ identity matrix.

Question B: Use the result of Question A to prove that for $A, B\in F^{n\times n}$, AB and BA have the same characteristic polynomial.

Question A is designed to evaluate basic formal and advanced representational understandings of matrix equivalence, and Question B – its applicational understanding.

Similarity

Formal understanding: the students can recall the definition of similarity.

Instrumental understanding: the students can implement the definition of similarity; they are aware that elementary operations do not preserve similarity.

Representational understanding: at the basic level, the students know that two diagonalizable matrices are similar if and only if they have the same characteristic polynomial and, in particular, the same determinant and trace. At a more advanced level they also know that triangulable matrices are similar if and only if they have the same Jordan form, and, in general, two matrices are similar if and only if they have the same rational form.

Relational understanding: at the basic level, the students know that similar matrices are equivalent. At a more advanced level, they also know that that similar matrices represent the same linear operator: if T is a linear operator on a finite-dimensional vector space V and if A represents T with respect to a basis α of V, and B represents V with respect to a basis β, then $B=P^{-1}AP$, where P is the transformation matrix from α to β.

Applicational understanding: the students know that similarity to a diagonal matrix (similarity to a matrix in a Jordan form) is very useful in differential equations, difference equations and computing polynomials of matrices.

The concept of similarity is a fundamental concept and thus it should be taught in all linear algebra courses. Ideally, the lecturer should aim at developing all types of understanding similarity in all students. The least demanding linear algebra course should aim at developing basic formal, instrumental, representational and relational understandings of similarity. In between the ideal and the least demanding scenarios, there is a room for the lecturers' trade-offs. The trade-offs may be decided upon accordingly to characteristics of the class, and it is important that the lecturer will be aware of the pedagogical consequences of the choices. For instance, for engineering and applied mathematics students the applicational understanding should be emphasized, and for pure math majors, for whom the course is an introduction to more advanced algebra courses and to functional analysis, achieving advanced level of representational and relational understanding of similarity is crucial. Another example of a trade-off is teaching the Jordan form but giving up teaching the rational form and, in this way, gaining some time for applications. This decision would be appropriate for courses of mixed audience. We are aware, of course, that in many cases the trade-off decisions depend on the teacher's preferences, and hope that this discussion may result in better grounded decisions.

Orthogonal Similarity

Orthogonal similarity has an important role in highlighting the connectedness of the course. For this reason, we focus here on the relational understanding of the concept. Relational understanding of orthogonal similarity presumes the knowledge that orthogonal similarity is both similarity and congruence, that it is a special case of unitary similarity and that two matrices are unitary similar if and only if they represent the same linear operator with respect to different orthonormal bases. In addition, orthogonal similarity is related, by means of Sylvester Inertia Theorem, to comparison of different methods of diagonalization, and, in turn, to Givens Method for computing the eigenvalues of real symmetric matrices.

DISCUSSION

Equivalence relations play an important role in mathematics, in general, and in linear algebra, in particular. Halmos (1982, p. 246) points out that the concept of an equivalence relation "is one of the basic building blocks out of which all mathematical thought is constructed". Skemp (1986) notes that the idea of equivalence relations helps to form a bridge between the everyday functioning of intelligence and mathematics. Many researchers and lecturers pointed out that constructing such a bridge is not an easy endeavour (e.g., Asghari & Tall, 2005; Chin & Tall, 2000; Chin & Tall, 2001; Mills, 2004), in particular, because the notion of equivalence relation is mathematically and epistemologically complex.

Stimulated by the famous paper "Understanding understanding mathematics" by Michener (1978), we try to understand and explain what it means to "understand" the concept of equivalence relations, and in particular equivalence relations between matrices. This is in line with and in continuation of the extended effort that has been made so far to promote students' conceptual understanding in linear algebra (e.g., Day & Kalman, 2001; Dorier, 2000, 2002; Dreyfus, Eisenberg & Uhlig, 2003; Harel, 2000; Jaworski, Treffert-Thomas & Bartsch, 2011).

In this paper we suggested a multi-facet definition of understanding equivalence relations between matrices and exemplified how the definition can be operationalized by means of mathematical tasks sensitive to its particular facets. We have also argued that an understanding of different types of equivalence relations between matrices, when taken as a central objective of a linear algebra course, can embrace most (if not all) topics usually taught in linear algebra courses. This also may educate the students to appreciate the applications of linear algebra and, at the same time, the mathematical structure and beauty of the subject. Consequently, the presented definition can be used as an organizational framework for planning, teaching and evaluating a linear algebra course.

More specifically, the choice of the equivalence relations, for teaching, depends on the level and the purpose of the course. Most one-year courses include row equivalence, column equivalence, matrix equivalence, similarity, congruence, unitary similarity and orthogonal similarity. A least demanding course can deal only with the first four relations, and a more advanced course can include also *congruence and consimilarity. The suggested definition of understanding equivalence relations may guide the lecturer in establishing feasible goals, in planning the course to achieve these goals and in evaluating the results. In some cases, a lecturer may be content with teaching aimed at particular types of understanding of the concepts at different levels of deepness. Reasons for this may be, for example, time constraints or the students' needs. Experienced teachers will decide what the minimum level of understanding of each type they want to achieve should be. We hope that our paper will help not only experienced teachers in doing the same. In addition, our operational definition may be useful as a part of a theoretical framework in a study dealing with students' learning of linear algebra or with development of lecturers' pedagogical and epistemological knowledge.

Although our paper is theoretical, we would like to mention here that we used the operational definition of equivalence relations as a framework for teaching a first year linear algebra course at the Technion. In addition, the tasks presented in the paper were tried in a series of informal interviews with the students who took part in the course and volunteered to participate in the interview. The interviews helped us to get an initial impression to which extent our conceptualization of understanding a concept is compatible with what students mean by understanding a concept and in particular, which types of understanding the concept of equivalence relations can be identified in students' mathematical performance. The interviews seem to confirm that the types and levels of understanding a concept of equivalence relations, described in the paper, are plausible. The validity of the suggested definition of understanding and the helpfulness of the suggested didactical approach will hopefully be tested in future empirical studies.

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