MULTIMODAL PROOF IN ARITHMETIC

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This theoretical paper develops further the concept of multimodal proof from the perspective of the multimodal paradigm, phenomenology and Luis Radford's theory of knowledge objectification. The study of such proof is motivated by its possible use in mathematics education, especially in school, but possibly also with adult students. We discuss one type of multimodal proof in arithmetic using a proof principle called schematic generalisation. It is argued that this type of proof both can establish truth in arithmetic and give phenomenologically explanations.

Key words: multimodal proof, embodied cognition, objectification, phenomenology

INTRODUCTION

The concept of multimodal proof was introduced in Rinvold and Lorange (2011). A multimodal proof is a generalized proof which beside written symbols and sentential reasoning can also include the visual modality, speech, the tactile and motor action, (p. 633). The idea of combining sentential and visual reasoning has been developed in mathematical logic and its learning by Barwise and Etchemendy (1996) under the name of heterogeneous reasoning. Their ideas have been used and followed up by several other researchers. An example is Oberlander, Monaghan, Cox, Stenning and Tobin (1999) who characterize heterogeneous proofs as multimodal. Several papers have followed up the multimodal perspective and Stenning and Gresalfi (2005) has applied heterogeneous reasoning to mathematics education, but none of these papers regard proof. We will now develop the concept of multimodal proof further within the multimodal paradigm of Arzarello and Robutti (2008). As far as we know, no other researchers have developed multimodal or heterogeneous proof within this framework.

The multimodal paradigm is an emerging view of thinking and reasoning. It combines the embodied mind paradigm and sociocultural theory. Mind is part of a physical body, and the cognising man acts physically and verbally in a physical and cultural world using artefacts and signs. Thinking is not only internalized speech, maybe supplemented by inner visualization, but is linked to all the senses and motor action. Thinking is made possible and restricted by our bodily life in the physical world, but has reached an advanced level through culturally developed language and artefacts. Multimodality is a direct consequence of this view of thinking.

It can be provocative to ask if proof could be generalized to include non-sentential modes of reasoning. The reason is the belief that proof is the core or heart of mathematics, and that proof equals formal sentential proof. Formal proof gives the subject its unique structure, precision and solidness and has been successful. Fallacies and idiosyncrasy have been a problem with visual and intuitive proof. However, we agree with research that opens for the possibility that some non-sentential proofs are legitimate.

> The mere existence of fallacious proofs is no more a demonstration of the illegitimacy of diagrams in reasoning than it is of the illegitimacy of sentences in reasoning. Indeed, what understanding we have of illegitimate forms of linguistic reasoning has come from careful attention to this form of reasoning, not because it was self-evident without such attention. (Barwise & Etchemendy 1996, p. 6)

We study communication and learning of proof by applying the theory of knowledge objectification, Radford (2006a, 2006b, 2008). The theory has its roots in Hegel and Husserl. Objectification has to do with the learning of the individual when thinking is seen to have an intimate and dialectical relationship with the material and cultural world, LaCroix (2012). It is a process using semiotic means in order "to draw and sustain attention to particular aspects of mathematical objects in an effort to achieve stable forms of awareness, to make apparent one's intentions, and/or to carry out actions to attain the goal of one's activity.", (ibid).

Our discussion will be restricted to one type of proof principle in arithmetic called schematic generalisation. We look at this through an example proof which is normally described as visual or diagrammatic proof. We argue that schematic generalisation can establish truth in arithmetic. These kinds of proofs have also been studied from the perspective of generic proof, which is a less precise concept. Referring to Tall (1979), Aliebert and Thomas (1991) write that "Such a proof works at the example level but is generic in that the examples chosen are typical of the whole class of examples and hence the proof is generalizable." Tall (1979) argues that generic proofs are explanatory in the sense of Steiner (1978), which writes that "It is not, then, the general proof which explains; it is the *generalizable* proof." (p. 144)". We argue that from the perspective of multimodal proof, proof by schematic generalisation are also explanatory in another way, which we call phenomenological explanation. A problem with many formal proofs, especially algebraic ones, is that the proofs are not explanatory. Students may be able to follow the rules which are applied in the proof, but they do not get any reason why the proved theorem is true.

WHAT IS MULTIMODAL PROOF?

A common view is that "the proof" can be separated from the activities of presenting a proof, finding or creating a proof, and the learning or understanding of a proof. Since we do not think "the proof" exists in a platonic world, it has to be physical or in the head of a person. The latter alone amounts to idiosyncrasy. From a sociocultural point of view "the proof" has to be an artefact which is accessible for public validation. In the rest of this paper proof is seen in this way. For a long period of time proof has been written or printed on paper. In this medium visual and sentential are the only possibilities. The existence of animation, film and video give a possibility of including other modalities, but also challenges the distinction between "the proof" and proof presentation. If the presentation of a proof is videotaped, the record is an artefact. We do not go further into this, but we discuss other modalities than the sentential and visual in proving and proof learning. Multimodality in the latter kind of activities is less controversial.

Multimodal or heterogeneous proving is a kind of multimodal reasoning and thinking. One core idea of multimodal thinking is that human reasoning always applies at least two modalities of thought.

> Multimodality, however, proceeds on the assumption that representation and communication always draw on a multiplicity of modes, all of which have the potential to contribute equally to meaning. (Jewitt, 2009, p. 1)

Except some proofs generated by computers, proofs are meant to be read by humans, and as such are part of communication. Because of this, words and visual diagrams are used together with mathematical formalism in the proofs. But, a common idea is that formal proof could be represented without visualisation by mathematical formalism only. The concept of visual proof indicates similarly the belief that visual arguments can be represented just by diagrams. The phrase "proofs without words", Nelson (1993, 2000), indicates the latter. According to Barwise & Etchemendy (1996), heterogeneous proof consists of more than one mode of reasoning, in their case primarily the visual and sentential modality. By proof those researches mean proof systems. Such a system consists of the allowed rules of inference and the allowed objects transformed by the inference rules. Since the formalism of Hilbert was developed, the objects of proof systems have mostly been formal sentences. The contribution of Barwise and Etchemendy is important for the question of legitimacy and possible acceptance of multimodal proof. Proof systems make validation of proof easier and also support the comparison with classical proof.

THE MULTIMODAL EXAMPLE PROOF

The example proof of our further discussion is given by a visual diagram and an explanation by word. As such it is multimodal, but it is open if it can be represented by a proof only in the visual mode. The proof is not formal, in the sense that it is not based on a proof system, and mathematical symbols are not applied.

Looking at the diagram in figure 1 below, it is not obvious what it is going to prove.

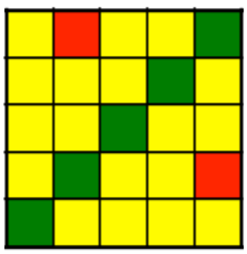


Figure 1

To be told that the statement concerns odd numbers may help some readers. The diagram shows the square of the odd number five. The diagonal in green consists of the same number of small squares as the side, and the red small squares above and below the diagonal make a pair.

Each small square in the upper triangle makes a pair with the corresponding symmetrically placed small square in the lower one. The complement of the diagonal thus is a set of disjoint pairs. Since the diagonal is a set of disjoint pairs together with a single small square disjoint from them, the large square is an odd number.

The argument shows that the square of an odd number is odd. But, we have more. The decomposition of the square into its diagonal and a pair of triangular numbers do not use that the side is odd, and hence is valid for all natural numbers. This can be used to show the opposite implication. If the square is odd, then also the diagonal is odd, for taking away a set of pairs from an odd number, results in an odd number. Since the diagonal equals the side, the side is odd when the square is odd. Formally the implications in both directions can be written

$$\forall x[\mathrm{Odd}(x)\leftrightarrow\mathrm{Odd}(x^2)]$$

RECURSIVE ω -PROOF AND SCEMATIC GENERALISATION

We argue that the proof principle used in the example proof can be formalized by the concept of recursive ω-logic, a proof principle which legitimacy hardly can be disputed. Beside this legitimacy argument, we also use ω-logic to make clear what the example proof is meant to exemplify. The origin of ω-logic is proof theory as a branch of mathematical logic, but the original use, called cut-elimination, is technical and outside the scope of this paper.

Jamnik, Bundy and Green (1997) introduced the formalization by recursive ω-logic for diagrammatic proofs like the example proof with the intention to argue that this kind of reasoning is legitimate. Each case of the theorem can be proved directly from a diagram by geometric operations. One given diagram plays a schematic role, which make it possible to generate the other diagrams and the proofs for each case. Those authors have developed a system for automated theorem proving called DIAMOND, which successfully have turned several diagrammatic proofs into recursive ω-proofs.

A recursive ω-proof of $\forall n\ \varphi(n)$ is a procedure which let us calculate a proof of $\varphi(n)$ for each n. In the example this means that we see the decomposition of the square as a procedure which can be done for all possible integer squares. Intuitively, this means that we can draw "the same kind of diagram" for all integer squares. In arithmetic $5^2 = 2 \cdot T_4 + 5$ is certainly not implying the general claim $n^2 = 2 \cdot T_{n-1} + n$, where T_4 and T_{n-1} are triangle numbers. What is different with the diagram is that it shows the decomposition to be more than an accidental identity between numbers.

A direct algebraic proof of φ(n) is also a recursive ω-proof of ∀n φ(n). The use of symbolic algebraic variables is based on some rules or properties which are common for all numbers in question, for instance the commutative and distributive laws in arithmetic. This is seen in the direct example proof **D1** of

$$\forall n[(n + 1)^2 = n^2 + 2n + 1]$$

The proof is given by

$$\textbf{D1}: (n + 1)^2 = (n + 1) \cdot (n + 1) = n \cdot n + n \cdot 1 + 1 \cdot n + 1 \cdot 1 = n^2 + 2n + 1$$

We normally see this as one proof of one conjecture, but an alternative is to consider it as a collection of proofs, one for each natural number. For instance the substitution of $n = 4$, gives a proof that $(4 + 1)^2 = 4^2 + 2 \cdot 4 + 1$.

Recursive ω-logic is an alternative to induction in the formalization of proofs in arithmetic. Indeed it is a stronger principle, because it implies induction in arithmetic. Induction is for each arithmetical formula φ the claim that φ(1) and ∀n[φ(n)→φ(n+1)] entails ∀nφ(n). We get an ω-proof from this by generating proofs of φ(n):

φ(2) is proved by φ(1) and φ(1) → φ(2)

φ(3) is proved by φ(1), φ(1) → φ(2) and φ(2) → φ(3), and so on.

Technically, induction proofs are a good solution, but typically such proof does not give an explanation understandable by students. As an example of the latter we prove the statement related to figure 1 by an induction proof **IN** setting φ(n) to be $n^2 = 2 \cdot T_{n-1} + n$, where $T_0 = 0$ and $T_n = T_{n-1} + n$ are the triangle numbers inductively defined. Using the statement proved in **D1**, we get

$$\textbf{IN}: (n + 1)^2 = n^2 + 2n + 1 = (2 \cdot T_{n-1} + n) + 2n + 1 =$$
$$2 \cdot (T_{n-1} + n) + (n + 1) = 2 \cdot T_n + (n + 1)$$

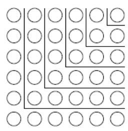
The case φ(1) follows also, as can be seen. The statement related to figure 1 can be proved by a direct algebraic proof **D2** too, but this appears as a rabbit thrown from a hat:

$$\textbf{D2}: n^2 = n^2 - n + n = 2 \cdot \tfrac{1}{2} (n - 1) \cdot n = 2 \cdot s(\mathrm{n}) + \mathrm{n}$$

Now, $s(\mathrm{n}) = \tfrac{1}{2} (n - 1) \cdot n$ is always a natural number, since $(n - 1) \cdot n$ has to be an even number.

PHENOMENOLOGICAL EXPLANATION

The phenomenology of proof has to do with how proof is experienced. The example diagram gives us the experience of knowing, understanding and believing. Jamnik, Bundy & Green (1997) formulate this about the same type of diagram,



$$1 + 3 + 5 + ... + (2n - 1) = n^2$$

“Not only do we know what the diagram represents, but we also understand the proof of the theorem represented by the diagram and believe it is correct (p. 51).” But, the diagram does not only give subjective belief. It is what Kitcher (1983) calls warranted belief (p. 17). The belief has to be justified in a way that is accepted by the mathematical society. Recursive ω-logic is one way of giving such a warrant, but an alternative place to look for it is embodied cognition. Arithmetic has a phenomenological and semiotic foundation which is profounder than axiomatic formalisations. According to Longo (2005), mathematics has cognitive roots:

> We cannot separate Mathematics from the understanding of reality itself; even its autonomous, "autogenerative" parts, are grounded on key regularities of the world, the regularities "we see" and develop by language and gestures.

This physical and perceptual basis of arithmetic can be used both to argue for the legitimacy and the experienced qualities of diagrammatic proofs. The legitimacy argument is to show that the kind of reasoning used in the diagrammatic proof is also needed to verify the formal axioms of arithmetic, but it is out of the scope and space of this paper to go further into this.

The possibility of arithmetic has to do with the stability of matter, that objects has permanence and do not suddenly appear, split or disappear like clouds. It also depends on our ability to discern some things as being a collection of objects of the same type. Freudenthal (1983, p. 75) points to Euklid book VII as an origin of the set or cardinal approach to number, and cites Felix Klein for the idea of numbers as collections of things of the same type. The concept of set is based on the invariance of physical or visual collections under spatial placement. The multimodal example proof is based on perceptible sets and spatial invariance. The objects of same type are small squares which together make up a square formed lattice. The decomposition of the square in the proof is related to spatial invariance, as it can be seen as moving the triangle parts away from the diagonal part. Both the concept of natural number and the proof also depends on our faculty of visual pattern recognition. That the proof uses the perceptual roots of arithmetic can thus be a reason behind its explanatory power. It is a phenomenological explanation not only by giving the experience of explanation, but also by using the phenomena behind the conjecture to be proved.

OBJECTIFICATION IN EMBODIED COGNITION

From the embodied mind point of view mathematics originates in our perception and ordering of physical reality. Even if the connection between parts of advanced mathematics and reality is not always obvious, this view makes it natural to look for reasoning with a perceptual basis, especially for the learning of the subject. But, we know from experience and research that students, or even mathematicians, are not able to immediately grasp the intended meaning of a proof from a diagram. This can be explained by the multimodality of thinking, that more than one modality is needed, but a semiotic approach gets deeper into the learning and communication aspect. Radford's theory of knowledge objectification is a theory about how individuals can be able to notice and make sense of what they do and see.

> ..., objectification becomes related to those actions aimed at bringing or throwing something in front of somebody or at making something apparent −e.g. a certain aspect of a concrete object, like its colour, its size or a general mathematical property. Now, to make something apparent, students and teachers make recourse to signs and artefacts of different sorts (mathematical symbols, graphs, words, gestures, calculators and so on). These artefacts, gestures, signs and other semiotic resources used to objectify knowledge I call semiotic means of objectification... (Radford, 2006b, p. 6)

The means of objectification are actions and semiotic resources. The example proof was given by a diagram and words. The words are semiotic resources which direct the attention of the reader to the appropriate aspects of the diagram. The concepts 'triangle' and 'diagonal' help the viewer to see the large square as composed of three parts. In the diagram also colours are used to make apparent the decomposition of the square into two triangle numbers and the diagonal, and also show how the small squares above and below the diagonal make pairs.

As an alternative or supplement, a physical diagram can be made of unifix or multilink plastic cubes. Then one of the triangular parts can be laid onto the other, both showing congruence and how to make pairs of cubes. We can pair a cube with the cube lying above it. This physical approach makes the red squares superfluous. These red squares are confusing as long as communication of the decomposition is in focus, so it would be an advantage if they could be painted yellow. Showing congruence does not mean proving, but pointing to. What looks like or feels like equality, can mistakenly be taken by the student to be a proof of equality. The physical process of making pairs of corresponding upper and lower small squares in the triangles, is a proof when the side equals five, but its generalisation requires an argument. A general proof requires another way of seeing.

The explanation by words introduces a process ordered in time, in which different aspects and parts of the diagram are in the foreground. Concepts like 'odd number' and 'set of pairs' helps the viewer to see the diagram as a general pattern. It is possible for a student to grasp everything else, but to see the diagram just as the case of five times five. It is a well known misconception among students that showing one or a few cases is enough to prove a result. We know that many mathematically trained persons experience to see a general proof through diagrams like the one in the example, but we also know that this does not come easy to many students. It is necessary to see the diagram like an informal ω-proof, that is, a procedure for generating the proof in all other cases. A possibility is to let the students draw and paint the five by five square and, notice how this is done and ask them to draw and paint squares of other sizes. Alternatively, the students can build squares by coloured plastic multilink cubes. The dynamic process of painting or building makes it more likely to see an algorithm than looking at a static diagram.

Husserl made a distinction between simple and categorial intuition. The latter means to 'intuite' the conceptual, the general or the Aristotelian form through seeing something concrete. According to Cobb-Stevens (1990), "Rather than presenting some particular thing, say a red chair, categorial intuition presents the chair's *being* red, the red quality's *belonging* to the chair (p. 44)." By intuition Husserl underlines the richness or fullness of actual experience compared to thought and speech. Cobb-Stevens exemplifies this as the difference between strolling through the streets of a foreign city and vague plans of a visit (p. 43). The grasping of the general in the multimodal example proof in visual or physical version probably has the same richness compared to the induction and direct algebraic proofs **IN** and **D2**.

ALGEBRAIC AND VISUAL PROOF COMPARED

The visual or physical multimodal proof related to figure 1 seems to work considerably better than the algebraic alternatives in order to give students meaning and richness of experience when understood or objectified. The former kind of proof gives at least another kind of explanation, which for many students probably is better. Since warrant for truth is the simultaneous establishing of truth and meaning, we think that even in this aspect multimodal proof is a good alternative. However, as some hints have indicated, objectification is not straight forward to achieve. For instance it can be difficult to see the conjecture to be proved directly from diagram 1, and seeing the diagram as general is demanding. Algebraic proof has some clear advantages compared to visual proof. Algebraic proof in arithmetic has well established proof systems which have an undisputed status among mathematicians. The system of algebraic and arithmetic signs are standardised, used almost everywhere and are institutionalised by schools, universities, books etc.

Like the visual diagram 1, also the arithmetical and algebraic notations are spatial and compact. The latter are not phenomenological or iconic, but symbolic. The signs together make a system giving meaning to terms and statements. Algebraic identities like $2x + 3x = 5x$ are linked to the objectified meaning of $2 + 3 = 5$ and the addition operator. Even the algebraic system has a link through objectification to physical and perceptual phenomena behind arithmetic, but not in the direct and full way as in categorial intuition. A relevant reference for the spatial aspects of algebraic symbolism is Bergsten (1999). As long as the complexity of an algebraic statement or a diagram is restricted, both have a good potential of objectification. Ordinary language composed by words lacks spatiality and compactness and are delegated to an intermediate role in learning and objectification. Proofs given by words are not alone a good way for students to objectify proofs. But, neither algebraic formulas nor diagrams give all the necessary information directly. Visual proof has to be supplemented by symbols, standards and transformation rules which make communication, objectification and validation easier.

CONCLUSION AND FURTHER RESEARCH

This paper has developed the concept of multimodal proof further by a discussion and clarification of what a multimodal proof is. One conclusion is that a multimodal proof is an artefact, but that the artefact of video challenges the distinction between proof and proof presentation and opens for other modalities than the visual and sentential. It is discussed how a type of visual proof involving a kind of schematic generalisation can be modelled and legitimated by recursive ω-logic. The learning and recognition of the generality of the proofs are studied by the theory of objectification. A new contribution is how building of physical versions of the diagrams makes the objectification easier by placing the procedural aspects in the foreground, giving a link to recursive ω-logic. The explanatory power of the visual proof has been discussed from the perspectives of phenomenology, categorial intuition and objectification. That the perceptual and cognitive aspect of the proof also is lying beyond the concept of number is suggested as a reason for the strikingly potential of explanation. A drawback of visual proof is the lack of standardisation, proof systems and semiotic signs making both interpretation and validation of the proofs easier. The development of these missing aspects both theoretically and by design is central in further research.

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