RESEARCH SITUATIONS TO LEARN LOGIC AND VARIOUS TYPES OF MATHEMATICAL REASONINGS AND PROOFS

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Abstract *– We present an analysis of the role of research processes and experimental activities in our "Research Situations", allowing students to learn fundamental knowhow in mathematics : experimentation, studying particular cases, reasoning, formulation of conjectures, examples and counterexamples, generalizing, proving, etc, as described in curricula and related documents of French junior high schools and high schools. The reasoning and proving processes are a full part of our SiRC. This will be illustrated here with examples.*

Key-words *– Research Situation, mathematics investigation, logic, reasoning, proof.*

INTRODUCTION

Routine researcher activities comprise elementary tasks such as : choosing a question, experimenting, studying special cases, choosing a solution framework, modelling, reasoning, stating conjectures, proving, defining, eventually changing the initial question, etc. *Knowledge and tools* to tackle these tasks are intrinsically part of a scientific approach, they are necessary to do mathematics and cannot be brought down to mere technics or methods.

In many countries, these basic tools are not available to a majority of scientific students. What they do and say when confronted with an « open problem » shows that their relation to mathematics is very far from the mathematician standpoint :

- « I don't know how to solve this problem, because it's new for me »
- « I don't know what to do, because I don't know the technique »
- « This problem is badly formulated, hypotheses are missing »
- « To which chapter is related this problem ? »

French mathematical curricula insist on experimentation, discovery and quality of scientific classroom activity for learning different types of reasoning and proofs, and involve some elements of mathematical logic. However, in schoolbooks and teaching practices, these objectives are not really treated, and the experimental approach and research activities are scarce. The objections raised by teachers to avoid the research situations in class are the institutional constraints and a lack of training to grasp and manage such situations. The idea that, in mathematics, one can experiment, model, study specific cases, infer conjectures from examples, is not present in the standard didactical procedures. Mathematics are seen as a service discipline, that offers a set of techniques and computation algorithms, although these can sometimes be sophisticated. Theoretical and fascinating aspects are kept inaccessible and reserved to experts.

The consequences of such positions and choices include attitudes and practices that seem to go against those which would precisely enable a significant research activity :

• Students are not allowed to change the hypotheses, nor to choose their own solution framework.

• Two strong « rules » become implicit, when students have to write or validate their proof :

« **Only** the problem data and the properties taught in class should be used in the proof »

« To complete a proof, one has to verify that **all** the hypotheses given are used »

• The mathematical activity is reduced to the technique of writing the proof, thus shrinking the research and arguing initial process.

HYPOTHESES TOWARDS GUARANTEEING A GENUINE MATHEMATICS ACTIVITY

H1. There is no real possibility of an investigation if a « toolbox » (theorems, properties, algorithms) is available and designated for the resolution of an obvious question. There is no real mathematical activity without a truth issue that can be taken and tested by students, while being non obvious to prove.

H2. It is neither necessary nor sufficient to arrange « real life » contexts (Coulange 1998) to perform a research or an experimental investigation : such contexts do not guarantee the relevance of the problem, and can even make noise that impede the investigation and the understanding of the underlying logic.

H3. It is not reasonable to propose research situations that bring into play mathematical concepts in construction. As a further advantage of avoiding such lapses, the problems will be accessible to many levels of knowledge (sometimes from primary school to university).

Our research work consists in conceiving appropriate specific situations – e.g. problems and didactical staging (in the sense of scenic design) – available in different institutional contexts, in experimenting them and analyzing their effects on the learning of mathematical reasoning, proofs, and the underlying logic.

CHARACTERIZATION OF THE MODEL “SIRC”[1]

This didactical model was already in gestation in Arsac & al. 1995 and Grenier & Payan 1998. Recall here the characterization of the SIRC model, as it has been described in Grenier & Payan 2003 & 2007. As any model, it is a reference (both epistemological and practical) for situations that we build, which can somewhat differ from the model.

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[1] SiRC : “Situation de Recherche pour la classe” , ie. “Research Situation for Classroom”

• A SiRC is similar to an *actual question* in mathematical research or in a non « didactified » one. This condition, very restrictive a priori, aims to avoid the question or the answer may seem obvious or familiar. The objective is to give relevance to the research activity. This condition can be artificially recreated by the « staging » of the problem.

• The initial question is easy to understand at various levels of knowledge. Our intention is to break with the usual didactic practice that tends to attribute any problem to a specific grade level. To fulfill this requirement, the statements must in general be not as heavily mathematized. However, we try to avoid random real life « noises », which complicate the task of students in non-mathematical « concrete » problems, and sometimes prevent them from entering into actual mathematics.

• Strategies to start with are available, but they won't solve the problem completely – usual techniques or properties are not sufficient. In other words, one must ensure the devolution of the problem, by leaving space to some uncertainty that cannot be reduced just by applying known techniques or usual properties (i.e., what Brousseau described in his theory as a « good » situation). The theoretical framework of resolution is neither given nor obvious.

• One can use several strategies, such as «trials and errors », study particular cases, etc ; relevant conjectures are not obviously true, counter-examples are attainable. These points are meant to encourage the construction of conjectures by students, based on an exploration of the question investigated. These conjectures can be examined by the students, through accessible examples and counterexamples.

• Hypotheses, or the initial question, can be changed. One can change the assumptions or the original question, and grab a new problem. The initial question can lead to related questions : closing a problem through the choice of certain parameter values, or starting a new research activity.

A SiRC is characterized by some *research variables* : problem's parameters which could be didactical variables (i.e. at teacher's disposal), but that are left at the student's disposal. The *social organization* is constitutive of a SiRC : working in small groups, material to play and search, time sufficient to search and discuss. Teachers dealing with a SiRC is a specific activity : training teachers is necessary.

Methodology and experimental results

The SiRC didactical model (Research Situations for Classroom) has been studied in the “Maths-à-Modeler” team for 15 years, using classical didactical theories and methodologies : construction of situations, experiments with numerous students at many different levels of knowledge (several hundreds of students) and analyses of the results. Some of these SiRC have been integrated in certain courses at university and in secondary schools. Results have been published in articles (Grenier 2002, 2003, 2008a, 2008b, Tanguay & Grenier 2009 & 2010) and in Theses (Ouvrier-Buffet 2003, Deloustal-Jorrand 2004, Godot 2005, Cartier 2008, Giroud 2011). We give in annex 1 a list of Research Situations for Classroom which have been studied for more than ten years.

Example of different types of reasonings and proofs worked in a SiRC

In the SiRC given in annex 2, « Tiling polyminoes with dominoes », in the case where the polymino is a square and $n = 3$, at every level of knowledge, properties and *conjectures* emerge from experimentations and research (Grenier 2008b):

> A *necessary condition* to tile a square, that is *not sufficient* – this result can be proved by a very easy *counter-example ;* Then, a *necessary and sufficient condition* to tiling emerges, according to the position of the hole. The proof consists in two steps : first, a *proof of impossibility* – by a « forced solution » tiling (*reductio ad absurdum*) – and a *proof of possibility* – by exhibiting an *example* of tiling (*existence property*).

Inductive and deductive reasonings and proofs in mathematics

Leading students to these admittedly different types of reasoning - inductive and deductive reasoning – is a declared goal of French college and high school programs. Generally, inductive reasoning aims to generalize to other objects a property known for certain objects, or to build new objects. Mathematical induction differs from induction in other sciences, especially in physics, by its intrinsic validity. Here is a well known excerpt from H. Poincaré :

> The induction applied to the physical sciences, is still uncertain, because it is based on the belief in a general order of the universe, an order which is always outside ourselves. Mathematical induction, namely, proof by induction, is required instead of necessity, because it is the assertion of a property of the mind itself. (H. Poincaré, Science and Hypothesis. Flammarion).

The activity of experimenting and studying special cases plays an important role, and is almost necessary in learning inductive reasoning, because it helps to establish and justify the formulation of conjectures – rather than raising them randomly, and helps to study these conjectures by going back and forth between the experimental data and the setting up of their evidence.

Reasoning by induction has the particular characteristic of being at the junction point of the inductive and deductive procedures. In my study of students and teachers conceptions on induction (Grenier 2002 and 2003), it appears that the understanding of the concept is severely lacking in depth. Induction is reduced to one or two techniques, and is perceived as a non-constructive tool of proof, which sometimes raises doubt for its justification. Accordingly, the scope of problems that can be solved by this tool remains extremely limited. Problems of various types designed to lead to a better appraisal of induction were introduced and studied (Grenier 2012).

A RESEARCH SITUATION INVOLVING OPTIMIZATION : HUNTING BUGS

The assigned task is to protect a grid field against “bugs”, by forbidding them to land on the grid. In order to do this, traps – uniminos – are available, each covering a box.

The bugs are small polyminoes (dominoes, triminoes, etc.), and they can "land" by covering exactly some boxes of the grid. The question is to find a minimal configuration of traps that protect the field. We consider in the sequel a 5 x 5 grid field.

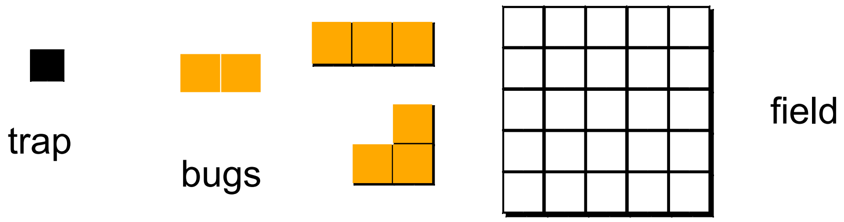


Figure 1

To solve the problem, students are equipped with appropriate material (wood, cardboard, etc.) that allows them to try and modify easily configurations without practical constraints. We are going to distinguish:

A « solution » : a set of boxes such that if one puts a trap on each of these boxes, then the field is protected. Putting a trap on each of the 25 squares is a trivial solution, but clearly a non optimal one.

An « optimal solution » : a configuration of minimum cardinality.

There may be several optimal solutions, namely sets of the same cardinality corresponding to different configurations.

The question is : for each of the three types of bugs represented above, what is the smallest number of traps that protects the area ?

Hunting Domino bugs

In that case, an optimal solution satisfies the following necessary condition : there are no two adjacent boxes without a trap. This condition is also sufficient : if there are not two adjacent boxes without a trap, then no bug can land on the field, because no domino can be put on the grid. Two « spontaneous » solutions satisfy this condition : one with 13 traps (figure 1a) and the other with 12 traps (figure 1b) – this property proves that the 13 traps solution is not optimal, but does not prove (yet) that the 12 traps configuration is optimal.

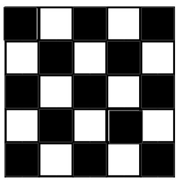


figure 2a

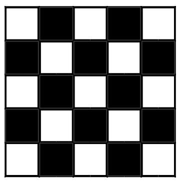


figure 2b

The following question is : Is it possible to protect the field with only 11 traps ? The answer is no. To prove this, we can study the « dual » problem, that is : what is the largest number of dominoes that can be placed on the grid area without overlap. Indeed, for a given pavement, it takes at least a trap by a domino-bug. We find easily that you can put down 12 dominoes without overlap, so at least 12 traps are necessary to protect the field. If the optimal number is denoted by N_{opt} it has been therefore established that :

12 traps are sufficient, that is, $N_{opt} \leq 12$

and 12 traps are necessary, that is, $N_{opt} \geq 12$.

So, we have proved that $N_{opt}=12$.

Numerous experiments of this problem, with students at different levels, show that a false property-in-act appears frequently, when the 13 traps solution is discovered first : « Any solution that is no longer a solution when an arbitrary trap is removed, is optimal ». figure 2b is of course a nice counter-example.

Hunting Long trimino bugs

Some of the reasonings and results that were established with dominoes can be reinvested here.

A solution satisfies the following necessary condition : there are no three adjacent boxes without a trap.

This condition is also sufficient : if there are no three adjacent boxes without a trap, then no bug can land on the field, because no long trimino can be put on the grid.

This second problem is more difficult than the first one : at every level, the experiments lead students to solutions that are far from the optimal one, such as the configuration given for the dominoes (figure 1b) – in a first step, a lot of students think that this is the optimal solution, because they cannot find any other. After performing new attempts, students frequently consider the one given by figure 3a below, with 9 traps. However, this configuration turns out not to be optimal. If no better solution is found, the teacher has to resume the situation, because otherwise students feel no clear incentive to continue the research.

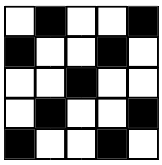
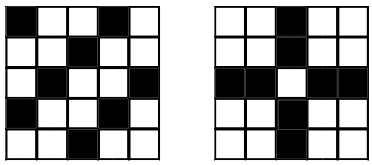


figure 3a. a 9-trap solution



figures 3b and 3c. two 8-trap solutions

Finally, there is almost always a group that finds one of the two 8-trap solutions (figure 3b and 3c). It remains to prove that these two solutions are indeed optimal. If one reinvests the « dual » proof already encoutered in the domino problem, the majority of students finds that it is possible to put down 7 long triminos without overlap (figure 4a), so at least 7 traps are necessary to protect the field. So, we have proved that $7 \leq N_{opt} \leq 8$. Finding a pavement with 8 triminos would allow them to close the question. However, this pavement seems in practice very difficult to find by students. The pavement in figure 4b proves that 8 traps are necessary to protect the field, that is $N_{opt} \geq 8$ (hence $N_{opt} = 8$).

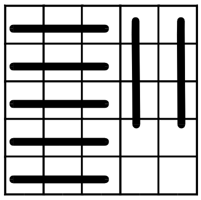
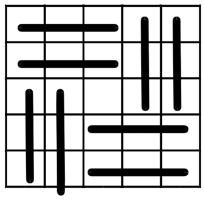



figure 4a **figure 4b**

Hunting L-trimino bugs

Reasonings and proofs to conduct in this third problem are more complex, essentially because of the less tractable shape of the triminos : it is not easy to give a necessary condition for a solution (that protects the field). It can be a good strategy to begin by finding a pavement of the grid by a maximal number of L-triminoes (dual problem). One can easily find that there exists a pavement with 8 L-triminos (figure 5). This pavement proves that $N_{opt} \geq 8$.

A recurrent false reasoning frequently follows this discovery : a claim that $N_{opt} \leq 8$, justified by the fact that it is impossible to put down more than 8 L-triminoes on the grid, as the equality $3 \times 8 = 24$ leads to 8 being the obvious maximum.

After a substantial time for experimentation and research, in general, many students find 12-traps solution (figure 6a), then 10-traps solution (figure 6b).

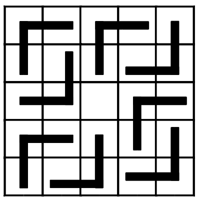
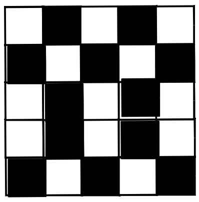
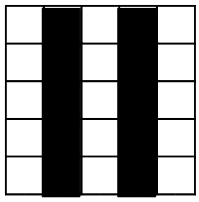




figure 5 **figure 6a** **figure 6b**

CONCLUSION

Learning argumentation, logical reasoning and different types of mathematical proofs requires work on appropriate specific problems, the resolution of which can be attained not just by applying techniques or formal results from the main course. We have built and experimented a number of "research situations" that should allow such acquisitions by students of various levels. We analyze especially a situation of "bug hunting" that enables students to construct conjectures and later to reformulate them in terms of implications, necessary and sufficient conditions (along with the methods of exhaustivity of cases, contraposition, reasoning by contradiction, counterexamples, etc). Our "SiRC" therefore contain all necessary ingredients to enter into genuine mathematical activities.

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ANNEX 1. SIRC EXAMPLES (MATHS À MODELER, IREM, UQAM)

Some examples of Research Situations for classroom, with the main concepts or tools that they bring into play.

Polyminos tilings /*algorithms, existence theorems* / graph theory

Hunting the bug / *optimization /* number theory, lower and upper bounds

Discrete geometrical objects / *representation, definition* / euclidian an non euclidian geometries

Moving on the discrete plane / *definition /* generating or minimal systems, linear algebra

Geometry at the mountain / *space representation /* non euclidian geometry, euclidian axioms

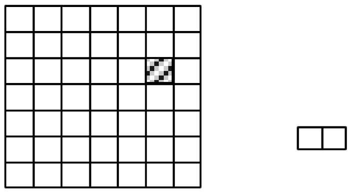
Regular 3D polyhedra / *defining, handling and proving /* 3D geometry

Regular polygons with integer vertices / *induction, ad absurdum /* combinatorial geometry

Disks in triangles or squares / optimization / combinatorial geometry, graph theory

ANNEX 2. TILING POLYMINOES

Problem P1. Rectangle with a hole in any position to tile with dominoes

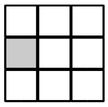


Some elements of the mathematical activity in the resolution of Problem 1

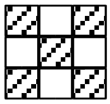
Properties and conjectures that emerge from experimentations and research

Property 1. A ***necessary condition*** to tile a square having a hole with dominoes, is that the area is even.

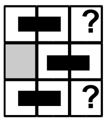
This condition is *not sufficient*. counter-example :



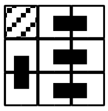
Property 2. For n = 3, a necessary and sufficient condition to tile with dominoes is that the hole is situated on any of the streaked position below.



proof of *impossibility* : by a « forced » tiling (ad absurdum)

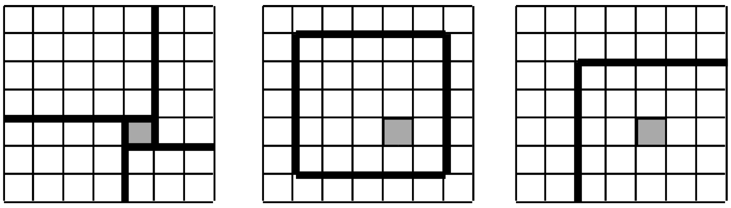


proof of *possibility* : by exhibiting a tiling (existence property)



These proofs are not transferable to any *n.*

Proof for every *n*, when tiling is possible : by structuration in rectangles with even area (without any hole), or by induction. This result is also true for rectangles



Proofs of impossibility for *n* arbitrary

Property. A necessary condition to tile is that, in a checkered coloration, the polymino is *balanced*

Proof by coloration, proof ad absurdum or by contradiction (if non balanced, then non tiling)

This condition is necessary for any type of polymino, but generally not sufficient.