0, 999….. = 1
AN EQUALITY QUESTIONING THE RELATIONSHIPS BETWEEN TRUTH AND VALIDITY

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The relationship between truth in an interpretation and validity in a theory as developed in logic, is a crucial issue in mathematics. In this article, we examine how an activity based on the equality "1 = 0.999 ..." could permit students to experience this relationship. We first provide practical and epistemological reasons supporting the claim that we have here a good candidate for this purpose. We then report on an experiment with nine fresh undergraduate students in France enrolled as volunteer in a short course aiming to help them to overcome difficulties in logic, reasoning and proof met in the calculus course; we analyse some excerpts of students' discussion showing that questioning this equality could favour the emergence of discussion on truth, validity, proof and theoretical reference.

Key words: truth, validity, decimal number, real number, limit.

INTRODUCTION

The relationship between *truth* and *validity* was first pointed by Aristotle who made it explicit in the First Analytics by introducing a clear distinction between de *facto truth* and *necessary truth*[1]. Modern logicians such as Wittgenstein (1922) and Tarski (1933) developed a semantic point of view in logic and provided a theoretical framework for the distinction between truth in an interpretation and logical validity, that concerns formal statements true whatever the relevant interpretation that is considered[2]. For example $[p \wedge (p \Rightarrow q) \Rightarrow q]$ is universally valid in propositional calculus (it is a tautology); that means that in any interpretation where p and q are interpreted by sentences A and B that are propositions[3], the sentence "*If A and if A, then B, then B*" is true, and this whatever the truth-value of A and B. Universally valid statements support *inference rules* that allow deduction in interpretations (Wittgenstein, 1922). The previous one supports the Modus Ponens, i.e. the well-known inference rule: ''*A*; and if *A*, then *B*; hence *B*''. While Wittgenstein restricted his approach to propositional calculus, Tarski developed it for quantified logic, by developing a semantics definition of truth that is *materially adequate* and *formally correct* (Tarski, 1933), through two crucial notions: the notion of *satisfaction* of an open formula by an element in an interpretation, and the notion of *model* of a formula. This last notion leads to the concept of *universally valid formula* as formula for which every relevant interpretation is a model.

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[1] For development on this point see for example Durand-Guerrier, 2008, p. 374.

[2] See Durand-Guerrier, 2008, pp. 376-377.

[3] In logic, a proposition is a linguistic entity that is either true or false.

It seems for us rather clear that understanding this distinction between *truth in an interpretation* and *logical validity* is a clue competence for proof and proving in mathematics: indeed, the correctness of a proof relies in an essential manner on the validity of the inferences that are involved in the proof. Moreover, it is clearly a challenge for mathematics education due to the fact that, as it is well documented in the literature, students often fail to understand why a proof is needed when they are convinced of the truth of a given statement. In this respect, it is necessary to introduce doubt in order to motivate argumentation and proof (Durand-Guerrier & al. 2012). Another important aspect in Proof and Proving in mathematics education is to make students aware that to provide a mathematical proof, it is necessary to work in a theory, at least a *local theory*.

It is well known in mathematics education that the equality “0,999…= 1” that appears when considering that the real number set coincides with the set of terminating or non terminating decimal number looks strange for a number of students, some of them considering that it is false (Tall, 2000; Dubinsky & al. 2005). Many reasons have been advanced for explaining these difficulties (difficulties taking in consideration non terminating decimal number; difficulties with limits considered as process, not as number etc..). In this communication, we document a new approach on these difficulties under the light of the distinction between truth and validity, and we aim to support the claim that working with this equality with undergraduate students may open discussion on this distinction with a benefice for both the mathematical and the meta-mathematical aspects (a better understanding of real numbers and an explicit example of the distinction between truth and validity). We first provide some *a priori* arguments supporting this claim; then we present some results of an experiment with fresh undergraduate students that we analyse through this lens.

SOME *A PRIORI* ARGUMENTS ON THE RELEVANCE OF QUESTIONING THE RELATIONSHIP BETWEEN TRUTH AND VALIDITY

For fresh undergraduate students in France, the familiar back ground for examining at first this equality is the terminating decimal numbers set, well known by students from elementary school, although it is generally not the case that they know well the specificities of this set among the others numbers sets. Anyway, they master the operative algorithms for sums, difference, product and decimal division; they have also met the case of those rational numbers whose decimal expansion has infinitely repeated sequence (repeating decimal). As a consequence, they have an empirical reference for the equality “1/3 = 0, 3333…. » (1), that is get by executing the algorithm for decimal division. By multiplying each side of the equality by 2, one gets a new equality “2/3 – 0,666….” (2); for this equality, there is also an empirical reference by making the division; this fact supports the conjecture that the calculating rules define on terminating decimal can be applied to non terminating decimal. But of course, it does not provide a proof that it is possible to extend the rule. In this respect, we have here a first aspect of the distinction between truth and validity. On the one hand, taking the result of the division algorithm as the definition of the decimal expansion of a rational number, then equality (2) is true. Accepting this equality is in general not problematic for students, but this equality could be perceived rather as the result of the process of dividing than as equality between two numbers. On the other hand, considering the set of decimal numbers, with addition, difference and multiplication algorithms on terminating decimal, it is not possible to deduce equality (2) in that theory using addition or multiplication by 2, due to the fact that we have change the nature of number and that we have no theory, even local, in which proving that multiplication could be extended. A fortiori, it is not possible to prove that we get a new equality by multiplying each side of equality (1) by 3. As a matter of fact, there is no empirical reference relying on division allowing considering the writing 0, 9999… as referring to a concrete process. At this point, arises the question of the possibility to find a theoretical justification of equality "0, 999…. = 1"

A natural candidate for justification: extension of operative algorithms

The extension of the operative algorithms from terminating decimal to non-terminating decimal seems to be the more natural candidate. We have seen that the empirical reference for equality (2) supports this conjecture. However, as said Dedekind in his correspondence with Lipschitz (Dedekind, 1876), it is not granted that the operation on integers or decimals should be extended to real numbers; this necessitates elaborating a theory. Dedekind did it by developing a construction of the real number sets with cuts, showing that the operation could be extended to the new set (Dedekind, 1872). Concerning non-terminating decimal numbers, there are some sound reasons for doubting of the possibility to extend the multiplication algorithm. There are practical reasons well known by students:

1. The algorithm for multiplication for terminating numbers is initiated on the right digit; in a non-terminating decimal number, where should we begin?
2. The product by 3 of a terminating decimal number that is not an integer is never an integer.
3. It is not possible to extend the algorithm for comparison.

There are also epistemological reasons:

4. As soon as one works with infinite, strange things may appear, so it is necessary to be cautious.
5. Does a non-terminating decimal writing refer to an object or to a process?

Point 2 pleads against the extension of the multiplication algorithm due to the fact that it leads to equality (3) that violates this theorem, and comforts point 4. Moreover, point 3 prevents an algorithmic comparison, which could be useful to decide if the two writings denote the same number. All of these aspects are likely to converge to rejecting equality (3). Point 5 open on a new question: how do we operate with process?

Process *versus* object – tend to *versus* have a limit

Point 5 could be seen as an inheritance of the dividing process; although this question has already largely been discussed in the mathematics education community (e.g. Cornu, 1981; Tall, 2000; Dubinsky & al. 2005), is still in debate, as it can be seen in a letter to Educational Studies in Mathematics' editors published on line on the website of the journal in 2011[4]. In this letter addressed to the editors the author considers that the following sentence out of a posthumous paper from Fischbein (Fischbein 2001) « students questioned whether 0, 333… is equal to 1/3 or tends to 1/3, answer usually that: 0,333…. tends to 1/3 which is not mathematically correct » should be corrected. The author of the letter argues that in mathematics the sum of infinite multiplicity cannot be equal to a number that is finite. Referring to the sum of the relevant geometric series, she concludes that the expression **"**0.333**…**tends to 1/3**"** will be mathematically true. The editor published the letter on line on 19 January 2011, and published on line a Reply on 01 march 2011 saying that

> "This letter to the editor was published online in the interest of open dialogue. However, the author's perceptions are not accurate, and this misconception was addressed already by Fischbein himself, and by other scholars who followed up on his work, e.g., by Dina Tirosh."

What interest us in this example is the fact that although the author of the letter refers to the theoretical point of view involving the sum of series, which defines the limit as a number that of course can be finite, the author considers that 0,333….. does not represent a number. We interpret this as an indication that the theoretical framework relying on sequences, series, limits and their operations could be insufficient to encompass the conception that such writing is referring to a process[5]. Notice that this point could be related to the distinction between potential and actual infinite.

Coming back to our equality "0,999…. = 1", such a position leads to reject this equality and to replace it by "0,999….. tends to 1":

> "In general, "limit" and "tends to" are not used in the same context. The limit designates something precise while it is possible to tend to something more vague. An example: we will say that the sequence " 0,9,0,99 0,999, 0,9999, … " has for limit 1" or "tends to 0,9999….." (…) For some students, an unlimited sequence has no limit … because it is unlimited. We observe that some students use the term "limit" for sequences whose limit is reached, and use the expression "tends to" when the limit is not reached". (Cornu, 1981, p. 325, our translation).

The author added that even among advanced students, the initial conceptions of limit are still present.

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[4] For the whole argumentation, see the « Letter to the editors ». http://www.springerlink.com/content/1l82j7w55h264602/

[5] This is also attested by blogs and forums that can be found on the web on this topic.

Truth *versus* validity

Relying of the previous considerations, we consider that this equality has a potential for questioning the relationship between truth and validity in the following sense: it is possible to consider that the equality is not true due to the fact that there is not an empirical reference allowing considering it as the result of a process.

The natural candidate for a theoretical framework (extension of algorithms for non-terminating decimal) is not satisfying (introduces a new result that does not suit with a solid result for terminating decimal). This is strongly related with the improper writing of integers and terminating decimal. The theoretical framework relying on limit of sequences could be insufficient for encompassing the common idea that 0, 999…. refers to a process rather than to a number. As a matter of fact, behind this lies the question of the definition of the notion of equality for real number: "two real numbers a and b are equal" if and only if "for every strictly positive real number ε, the distance between a and b is strictly inferior to ε", that is beyond algebraic calculation and classical comparison algorithm. On another hand, it is possible to consider that this equality is true and to be convinced by arguments that could not be accepted as proof, in particular because they rely on the extension of operative algorithms.

In the second part of the paper, we report on an experiment with fresh undergraduates students in France (Lyon, January 2008) where the discussion opened on some of these questions.

FRESH UNDERGRADUATE STUDENTS STRUGGLING WITH TRUTH AND VALIDITY

The experiment that we report here took place in Lyon (France) in January 2008 with nine first year university students (three girls and six boys). They had followed in fall semester a calculus course. They were volunteers for following a short course (eighteen hours in three days) aiming to help them to overcome difficulties in logic, reasoning and proof met in the calculus course. Seven of them were fresh students facing difficulties; one was in reconversion after having prepared during two years the medicine competitive exam; the last one was a very good student who wanted to deepen his logical and proof and proving competencies. We focus on a session that took place on day 2 and was devoted on proof and more precisely on the work around the equality « 0,9999 … = 1 ». The students were asked whether the equality were true.

Possible students' answers to the question

The question is to evaluate equality were one side refers clearly to an integer, while the second side is a writing referring to a non-terminating decimal "number". As said in previous paragraph, the nature of 0,999…is not obvious. Theoretically, non-terminating writings refers to real numbers, while empirically, in some cases, it refers to process; the specificity of 0,9999… is that it is difficult to imagine a concrete process leading from integers to such writing, so that there is no available empirical argument supporting the equality. We list below the classical answers that we should expected

***Answering NO*, with various justifications**

N1: The two numbers are different in nature: one integer and a repeating decimal

N2: That is obvious

N3: The whole parts are different

N4: It is always possible to add a 9 to the sequence

N5: 1 is the limit; it is not reached.

N1 refers to separate classes of numbers while the theoretical point of view consider that integers belong to the real number set. N2 corresponds to identification between *form* and *object*, while in a theoretical point of view different writings may refer to the same object. N3 correspond to an application of a rule valid for terminating decimal expansion, but not for non-terminating one. N4 and N5 could be interpreted as the consideration that the writing 0, 9999….. represents a non ending process (potential infinite) while the theoretical point of view acknowledge that this writing refers to an object (actual infinite). .

***Answering Yes, with various justification*:**

Y1: I know it because I have learned it.

Y2: "1 = 3 ×1/3 = 3 × 0,3333…" This uses the implicit extension of the algorithm of multiplication for terminating decimals to non terminating decimals. As we have said, this arises the question of the validity of this extension.

Y3: Using a classical technics for identifying the rational number associated to a periodic decimal expansion**.** Let $a = 0{,}999\ldots$; multiplying a by 10 gives $9{,}999\ldots$; subtracting a to 10*a* gives 9 *a*; hence 9*a* = 9 and then *a* = 1. Once again, this technic relies on the extension of the algorithms of multiplication, and also of subtraction, that should be questioned. In this respect, it is not a proof.

Y4 Showing that for every positive real number ε, $|1- 0{,}999\ldots| < \varepsilon$. This could lead to the following justification using a geometric sequence, or a geometric series.

Y5: Considering 0,9999… as the limit of the sequence u_n = 0, 999…9 with n digits 9, for n≥1, and then show that the limit is equal to 1.

Y6 Considering 0, 9999… as the limit of the series $\sum u_n$ where $u_n = 0{,}9\times10^{-n}$. The infinite sum $\sum_{n=1}^{+\infty} u_n$ is a theoretical mean to express the decimal expansion 0,9999….; it is possible to prove that the series converge to 1.

Y1 corresponds typically to the consideration that things are true because they are said to be true. This is precisely something we should like the students to overcome during the short course. Y2 and Y3 correspond to the justification of algebraic type relying on the extension of algorithms. Y4 is a theoretical justification, whose validity is proved in the theory of real numbers, but it is difficult to use it without introducing sequences. Y5 and Y6 rely on the theoretical framework of numerical sequences and series. Their introduction along with the notion of limits, the operation on the sequences and their compatibility with limits, and the uniqueness of limit allow to provide a proof of the studied equality.

According to us, this *a priori* analysis enlightens the fact that debate on truth and validity is likely to emerge from the discussion. We present now selected excerpts of the exchanges in the different phases of the session; we will use our classification of answers and justifications as a grid for our analyses.

Analyse of students' exchanges

Students had to decide if the sentence was true or not; they worked individually for 10 minutes; then they were invited to present their production in front of the group. The session has been led by the second author of this paper; the first author attended the session as observer, took photos and audio recorded the debate. We point now some phenomena that have been observed.

"1 is equal to 1, and nothing else"

Student G1 rejected the equality, insisting that

9-G1: "1 is equal to 1 and nothing else"

We could interpret this answer as asserting a *material adequation*. Indeed, during the individual work, this student had said that: " It sees itself". Another student F1, that had at first answered: "Yes", seemed convinced by G1 argument:

11-F1: I said yes because I was once told that it is true, but I agree with G1, it is very disturbing. I said yes because it's said to me.

Her trouble seemed to come from the fact that she knew the sentence is true, but she did not know why. Opposite with G1, she did not explicitly reject the equality.

A tentative to prove

Student G2 proposed a theoretical justification of its answer, trying to provide the " epsilon proof " relying on the equality between real numbers, which corresponds to the justification Y4.

16 G2 In fact I have said yes, but I went to euh, with a positive epsilon. In fact, the euh only, the only "1 – 0,999 ..." which is less than epsilon is zero.

18 G2 If we find one "1 - 0,999" which is lower than an positive epsilon; we consider that 0,99 is fixed, finished.

26 G2 We know that 0.999 .... It is an infinity of 9 after zero, after the comma. But if "1 - 0.9999 ..." is less than epsilon, 0.9999 ... is finite, which leads to an absurdity because in the beginning, it is assumed that it is. .. it is assumed that it is an infinite number..

The tentative did not convince other students who address remarks to G2:

E: "I don't know why absurd is! "

E: " I don't see what is the used of epsilon!'

G2, **G3** and **T** try to clarify the project of **G2; G3** introduced the sequence $1 - u_n$ for which the sequence 10^{-n} is an upper bound. The explanation becomes clearer to others, but *epsilon* disappeared. **G2** goes back to his demonstration, reintroduced epsilon, trying to clarify its role, and explaining that he planned to show that 0, 9999… is not different to 1 (proof by contradiction).

93 G3 Let us look for an epsilon such that the difference is greater than epsilon.

94 T So if we want to show that 1 is different from 0, 999…., we should show that we can find an epsilon such that 1 – 0,999….. is greater than this epsilon.

95 G2 This is what I wanted to do

97 G2 In fact, just to assume that it is different and show the opposite.

G2 finally did not managed to clearly expose his proof, the technical steps remaining confuse. As we said in the *a priori* analysis, the proof with epsilon is difficult if the sequences are not introduced. The other students give up following him and turn to other proofs.

"Equal to the limit versus tends to the limit"

Later in the session, a student proposed the proof *Y5*, and another one tried without success, to implement the proof *Y4*.

While she recognizes that proof *Y5* is clear, F1 is still not convinced; she engaged in a discussion with the teacher on the difference between "to be equal to the limit" or to "tend to the limit"; considering that the limit is not reached, she asserts that the proof "does not prove", and finally staked anew that she knows that it is true, but she does know why.

110 T: For you, 1 is 1. You are right, but what about the limit of u_n for you ?

111 F1: It tends to 1! It tends only; it is not equal.

112 T: No, it is the sequence that tends.

113 F1: yes

114 T: But the limit?

115 F1: It is not reached

127 F1: Ben? Then, we proved nothing?

134 F1: That approaches

141 F1: I know that it is yes, but I don't understand

This position of F1 is in line with the results of Cornu (1981) and makes an echo to the position of the author of the letter to editor we presented in paragraph I.

Extension of operations on non-terminating decimal numbers

T intervenes to ask F1 providing another writing of 1/3. She gave 0.333 ... with infinitely many digits equal to 3; this does not disturb nor shock her, she argued that it is different: **F1 157** It is not the same thing, because if you say 1/3, it is a finite number

Then she said

163 F1 Ah, but if you multiply everything by 3, it runs, it gives the same thing here.

At that moment, F1 seems to become convinced.

We should consider that for F1, the facts that "There is an empirical evidence for the decimal expansion of 1/3", and "Multiplying both sides by 3 provides the equality" are solid enough to ground the truth of the equality, answering her « why » pending interrogation, this although the extension of multiplication to non-terminating decimal number has to be established.

A discussion on proof Y3

Students F2 who arrived while the discussion was already engaged (round 144) asserted that the equality is true and that it can be easily proved. She proposed the proof Y3.

The teacher asked F2 if multiplying by 10 the decimal expansion is allowed. F2 then move to proof Y2, but the teacher asks again if it is possible to extend the operations; finally F2 gives up. The intervention of F2 and the teacher question introduces anew in the debate the discussion on truth *versus* validity.

178 T: [...] " What allows us to still apply the operations while we have no more the process of division? For 2/3, we had a process of division?

183 F2: We can also demonstrate by 1/3. 1/3 is 0, 333…; 2/3 is 0, 666...; 3/3 is normally 0,999…

185 F2: We have no right

At the end of the session, the student F1 remains disturbed by the equality; this leads her to accept the proofs while remaining sceptical on the truth-value of the equality.

221 F1: It always disturbs me

225 F1: It is good the proofs, it is attractive. But it is proved OK, but "1 is 1".

F1 seems to consider that the statement could be proved in a theory, while it would be false in a given relevant interpretation.

CONCLUSION

In this communication, we intend to show that working on the equality 0,999… = 1 is likely to open rich discussion on the relationships between truth in an interpretation and logical validity of a proof in a theory. The exchanges reveal three main attitudes: knowing that the equality is true and knowing that there is a theoretical proof, even if exposing the proof is difficult – knowing that it is true and knowing a proof relying on the extension of operations to non terminating decimals, without having a proof that this can be done – knowing that it is said to be truth, but without knowing why – understanding a proof, but remaining doubtful concerning the truth of the equality, opening the possibility of a discrepancy between theoretical and empirical assumptions. It is more usual for students to be sure that a statement is true, and do not understand the need for proof. In the case we have presented, it seems that students express a need to proof that implicitly engaged toward a need for theory. These results confirm our hypothesis that we have here a good candidate for discussing these topics, but it is clear that going deeper on these questions with students necessitate to design more cautiously a didactical situation, in order to allow students

> “ (…) to experience not only how to validate statements according to specific reference knowledge and inference rule within a given theory, but also how the “truth” of statements depends on definitions and postulate of a reference theory.” (Durand-Guerrier & al. 2012)

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