THE MAKING OF A PROOF-CHAIN: EPISTEMOLOGICAL AND DIDACTICAL PERSPECTIVES

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We will present an epistemological narrative based on a corpus of historical texts pertaining to the theorem: the sign of the derivative of f determines the variations of f. From a didactical perspective, the main points are: (1) the texts illustrate the role of "local" counter-examples (to use Lakatosian terminology); (2) the various proof-attempts are based on at least two pretty different proof-ideas; (3) even a proper (meaning, both intuitive and formal) understanding of the concepts involved in the statement of the theorem may lead to a faulty proof scheme; (4) it helps understand how long deductive chains emerge and stabilise. On the basis of this narrative, we eventually underline connections with current research works on proof in mathematical analysis, and mention teaching and teacher-training perspectives.

Key words: epistemology, proof analysis, mathematical analysis, AMT.

RATIONALE

In the twentieth century, most tertiary-level textbooks of mathematical analysis prove the following theorem: let *f* be a differentiable real-valued function defined on an interval, if its derivative *f'* is positive, then *f* increases over this interval. Its standard proof is a rather straightforward application of the "mean value theorem"[1] ("*égalité des accroissements finis*" in French, "*Mittelwertsatz*" in German); the proof of which is a rather straightforward application of the "Rolle theorem"[2], which, in turn, depends on the fact that a continuous real-valued function defined on a closed and bounded (i.e. compact) interval has a maximum or a minimum. The latter fact, although quite intuitive, depends on not-so-trivial properties of the set of real numbers (completeness of the metric space, local compactness). Historically speaking, this *deductive chain* can be found in the textbook (Jordan, 1893, p.65-67).

With this example, we can see that the proof of a rather intuitive qualitative fact (namely: if all the tangents point upward, the curve has to move up) requires several layers of sophisticated concepts (differentiability, continuity, properties of the numerical continuum), and a few standard tricks (affine changes of variable). In this

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[1] Let *f* be a differentiable real-valued function, defined over some interval [*a*,*b*], there exists a value *c* between *a* and *b* such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. Geometrically speaking: on the arc of curve joining the points (*a*,*f*(*a*)) and (*b*,*f*(*b*)), there is a point where the tangent is parallel to the chord joining the two endpoints.

[2] Let *f* be a differentiable real-valued function, defined over some interval [*a*,*b*], such that *f*(*a*) = *f*(*b*). There is a value *c* between *a* and *b* for which the derivative vanishes.

paper, we will present some of the proofs given, over the 19th century, either of this mathematical fact, or of some key points in its proof; and some instances of critical proof-analysis.

We must stress the fact that this paper is not a work in the history of mathematics, but a work of an *epistemological* nature *based* on a historical corpus. The corpus consists of documents that we selected and translated into English (from the French or German languages). For lack of space, only part of the corpus can be presented here (for a more comprehensive presentation, see (Chorlay, 2012)). Quite a few works in maths education research have focused on similar issues (for recent examples: (Arsac & Durand-Guerrier, 2005), (Barrier, 2009)): our goals are (1) to make this new corpus available to this community of researchers; (2) to compare what this corpus helps document with current research perspectives on proof in mathematical analysis; and (3) to point to potential uses in a teaching or teacher-training context.

LAGRANGE'S PROOF (1806)

Let us quote the beginning of Lagrange's proof

> *A function which vanishes when the variable vanishes, will, as the variable increases positively, have finite values of the same sign as that of its derived function; or of the opposite sign if the variable increases negatively, as long as the values of the derived function keep the same sign and do not become infinite.*
>
> (...) Let us consider the function $f(x+i)$, whose general development is
>
> $$f(x) + if'(x) + \frac{i^2}{2!}f''(x) + \cdots .$$
>
> As we saw in the former lesson, the form of the development may be different for some specific values of *x*; but we saw that, as long as $f'(x)$ is not infinite, the first two terms of the expansion are exact; and that the other terms will, consequently, contain powers of *i* greater than the first, so that we shall have
>
> $$f(x+i) = f(x) + i[f'(x) + V],$$
>
> V being a function of *x* and *i*, which vanishes when $i = 0$.
>
> So, since V vanishes when *i* vanishes, it is clear that, should *i* be made to increase from zero through imperceptible degrees, the value of V would also increase from zero by imperceptible degrees, either positively or negatively, up to a certain point, after which it may decrease; consequently, one will always be able to assign to *i* a value such that the corresponding value of V – regardless of the sign – is less than any given quantity, and that for lesser values of *i*, the values of V are also lesser.
>
> Let D be a given quantity, which may be chosen as small as one pleases; one can always assign to *i* a value so small that the values of V are bounded by the limits D and –D; so, since we have
>
> $$f(x+i) = f(x) + i[f'(x) + V],$$

It follows that the quantity $f(x+i)-f(x)$ will be bound by these two

$i[f'(x) \pm D]$. (Lagrange, 1884, p.86-89)

In this passage, we can see that Lagrange also had a proper numerical understanding what the value of the derivative at a given point represents, and that he did interpret limits as relationships of dependence between inequalities. For instance, he rephrased "V being a function of x and i, which vanishes when $i = 0$" as "one will always be able to assign to i a value such that the corresponding value of V – regardless of the sign – is less than any given quantity (...)".

In the part of the proof which we omitted (see (Chorlay, 2012) for a more comprehensive translation), Lagrange applied the above inequalities for x-values of type $x+i$, $x+2i$, ..., $x+(n-1)i$, determined an upper bound for the sum, then passed to the limit. In spite of the fact that the theorem Lagrange set out to prove is correct, and that the proof relied on a correct numerical understanding of the derivative construed as a limit, something does not sound right in the proof: a critical reader may spot hidden uniformity assumptions (namely: uniform derivability), and circular chains of dependent quantities (see (Barrier, 2009) for a detailed analysis of very similar cases, and (Ferraro & Panza, 2012) for recent historical work on Lagrange).

CAUCHY'S PROOF (1823)

Problem. Assuming that the function $y = f(x)$ is continuous relative to x in the neighborhood of specific value $x = x_0$, one asks whether the function increases or decreases as from this value, as the variable itself is made to increase or decrease.

Solution. Let Δx, Δy denote the infinitely small and simultaneous increments of variables x and y. The $\Delta y/\Delta x$ ratio has limit $dy/dx = y'$. It has to be inferred that, for very small numerical values of Δx and for a specific value x_0 of variable x, ratio $\Delta y/\Delta x$ is positive if the corresponding value of y' is positive and finite. (…)

This being settled, let's assume function $y = f(x)$ remains continuous between two given limits $x = x_0$ and $x = X$. If variable x is made to increase by imperceptible degrees from the first limit to the second one, function y shall increase every time its derivative, while being finite, has a positive value. (Cauchy, 1823, p.37)

Unlike Lagrange, Cauchy defined the derivative as a limit; just like Lagrange, he was able to derive proper numerical conclusions from this numerical conception of the derivative. So what makes his argument so different from Lagrange's ? Actually, they do not have the exact same understanding of what the *conclusion* to be reached is; both have implicit definitions[3] of what it is for a function to be increasing, but their definitions do not match exactly. Lagrange's definition is closer to the one we find in today's textbook: a real valued function defined over some interval I is an increasing

[3] We could use the notion of "in-action definitions" (Ouvrier-Buffet, 2011). For a more detailed analysis of Cauchy's proof, see (Chorlay, 2007).

function if, a and b being any elements of I, $a < b$ implies $f(a) < f(b)$. Lagrange's (implicit) definition reads slightly differently, since he compared the values of f at 0 and at any other *given* value.

Cauchy's implicit definition of an increasing function can be rephrased as follows: a real-valued function f defined over some interval I is an increasing function if, a being any element of I, there is a neighbourhood N_a of a such that, for any x in N_a, the order between $f(a)$ and $f(x)$ is the same as that between a and x. Lagrange's definition is global, point-wise, and refers to two (arbitrarily, independently) given points; Cauchy's definition is one in which some local property holds in the neighbourhood of every (arbitrarily) given point. It can be shown – but it takes a little work – that both definitions are actually equivalent from a mathematical viewpoint[4]. However, they differ significantly, both from an epistemological viewpoint (in which, for instance, the difference between local and global properties are put to the fore), and from a cognitive viewpoint (Chorlay, 2007).

The fact that both definitions coincide from a mathematical viewpoint does not imply that proving that the first holds involves the same kind (and amount) of work than proving that the second holds. The information we start with (sign of the derivative) being of the everywhere-local-type, a mere rephrasing of the hypotheses leads to Cauchy's definition of increasing functions, hence to the conclusion. Reaching Lagrange's conclusion involves patching up local pieces of information to reach global conclusions, an endeavour which the modern reader knows to be usually tricky.

BONNET'S PROOF (IN J.-A. SERRET'S TEXTBOOK, 1868)

Theorem I.- *Let f(x) be a function of x which remains continuous for values of x between two given limits, and which, for these values, has a well-determined derivative f'(x). If x_0 and X denote two values of x between these same limits, the following* $\frac{f(X)-f(x_0)}{X-x_0} = f'(x_1)$ *, will hold, with x_1 a value between x_0 and X.*

Indeed, the ratio $\frac{f(X)-f(x_0)}{X-x_0}$ has, by hypothesis, a finite value; and, if A denotes this value, we will have (1) $[f(X) - AX] - [f(x_0) - Ax_0] = 0.$

Let $\varphi(x)$ denote the function of x defined by the formula

(2) $$\varphi(x) = [f(x) - Ax] - [f(x_0) - Ax_0],$$

then, from equality (1), $\varphi(x_0) = 0,\ \varphi(X) = 0,$

[4] One must nevertheless stress the fact that if the Cauchy property holds at one given point $x = a$, it does not imply that the function is increasing in any neighbourhood of a. Consider $x + 10\,x^2 \sin 1/x$ in the neighbourhood of 0.

so that $\varphi(x)$ vanishes for $x = x_0$ and for $x = X$. Let us assume, for instance, that $X > x_0$, and let x increase from x_0 to X; at first, the value of $\varphi(x)$ is zero. If we assume that this function is not everywhere zero, for values of x between x_0 and X, it will have to either begin to increase, thus taking on positive values, or begin to decrease, thus taking on negative values; be it from $x = x_0$, or from some other value of x between x_0 and X. If these values are positive, since $\varphi(x)$ is continuous and vanishes for $x = X$, it is obvious that there will be a value x_1 between x_0 and X such that $\varphi(x_1)$ is greater than or equal to the neighbouring values $\varphi(x_1 - h)$, $\varphi(x_1 + h)$, h being an arbitrarily small quantity.

[Serret then proved that $\varphi'(x_1) = 0$, by a well-known argument; and stressed that this proof idea is Ossian Bonnet's]

(...) Theorem III.- *If the derivative f'(x) of function f(x) remains finite for all the values of x between the limits x_0, if X > x_0, and if x is made to increase from x_0 to X, the function f(x) will increase as long as the derivative f'(x) will not be negative, and it will decrease as long as f'(x) will not be positive.*

Indeed, since x lies between x_0 and X, the ratio $\frac{f(x \pm h) - f(x)}{\pm h}$ has limit $f'(x)$, which is a finite quantity; so it will of the same sign as that of the limit, for values of h between zero and some sufficiently small positive quantity ε. Consequently, for these values of h, the following will hold

$$f(x-h) < f(x) < f(x+h) \quad \text{if } f'(x) \text{ is } > 0,$$

$$\text{and } f(x-h) > f(x) > f(x+h) \quad \text{if } f'(x) \text{ is } < 0.$$

Thus, the function $f(x)$ will increase, as from any value of x for which $f'(x)$ is > 0; and decrease, as from any value of x for which $f'(x)$ is < 0. (Serret, 1900, p.17-22)

In this passage, Serret introduced Bonnet's proof of the mean value theorem, a proof idea which relied on an affine change of variable and the vanishing of the derivative at a local extremum. The existence of the extremum is not proved (at least when one compares with later rewritings of this proof), but made obvious in the *narrative* style which is so typical of the first half of the 19th century.

Strikingly, Serret did not use the mean value theorem to establish the relationship between the sign of *f'* and the variations of *f*; he relied on Cauchy's argument, hence on Cauchy's notion of functional variation.

PROOF-ANALYSIS AND REGRESSIVE ANALYSIS

Proof-analysis: the role of uniform convergence

We identified in Lagrange's proof a flaw which can be described in several ways: implicit assumption of uniform differentiability; failure to notice that some variable is dependent on some other, while trying to consider the limit of second while leaving the first fixed; exchange, without due caution, of two limiting processes. The same flaws were common to most proofs in analysis which dealt with the numerical aspect of functions (as opposed to formal aspects) (Dugac 2003) (Chorlay 2012).

At this point, one could choose to focus on texts where the new concepts of uniform convergence/continuity were first expressed with full clarity (Dugac 2003). Instead, we would like to stress the interest of texts which criticized faulty proofs, or spotted hidden lemmas. For now, let us quote just one excerpt from the correspondence between Hoüel and Darboux[5]. In this passage (dated Jan. 1875), Darboux comments on the standard argument he read in the drafts of Hoüel's textbook:

> Here is where I find fault with your reasoning, which no one deems rigorous any more. When setting
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> $$\frac{f(x+h)-f(x)}{h} - f'(x) = \varepsilon,$$
>
> ε is a function of the two variables x and h which tends to zero when, leaving x fixed, h vanishes. But if x and h vary, as in your proof; even more, if every new subdivision $x_1 - x_0$ generates new ε quantities, I cannot see anything clearly any more, and your proof becomes only seemingly rigorous. (...) You could get out of this predicament in one of two ways, 1. By changing proofs altogether, which I advise you to do. 2. By proving that if a function always admits a derivative between x_0 and x_i, one can find a quantity h such that for all values of x between x_0 and x_i, and all values x_0 and h_1 of h less than some limit value, one has
>
> $$\frac{f(x+h)-f(x)}{h} - f'(x) < \varepsilon,$$
>
> where ε has a value which is fixed but chosen as small as one wishes; which is difficult[6]. (Gispert, 1983, p.99-100)

Regressive analysis: the role of the existence theorem for extrema

A critical mind might object to Bonnet's proof of the mean value theorem that it depends on the existence of a maximum or a minimum, an existence which is implicitly taken for granted. It seems clear that if the function is piece-wise monotonous (as seems to be assumed in the text), it will indeed admit either a local maximum or a local minimum; but a differentiable function needs not be piece-wise monotonous, as the ever useful example $f(x) = x^2 sin\frac{1}{x}$ shows.

In fact, the existence of a maximum can be grounded without piece-wise monotony, or continuous differentiability, as Weierstrass established, for instance in his 1878 lectures on the theory of functions. The following passage has nothing to do *a priori* with calculus. It comes after the construction of the set of real numbers **R** (or, more

[5] This correspondence was published in (Gispert, 1983). It was discussed in (Balacheff, 1987).

[6] The fact that, if the derivative is continuous, then $\frac{f(x+h)-f(x)}{h}$ *does* tend to $f'(x)$ *uniformly* on every closed and bounded interval was proved, for instance, in the second edition of Jordan's textbook (Jordan, 1893, p.68).

precisely, $\overline{\mathbf{R}} = [-\infty, +\infty]$) starting from rational numbers. Weierstrass derived the existence theorem for extrema as a consequence of a very general and abstract theorem, proved using nested intervals (we quote the theorem but skip the proof):

> Let a value y correspond to every point $(x_1, \ldots, x_n)$ of some domain; then y is also a variable quantity, hence has a lower and an upper bound; let g denote it. Then, there exists at least one point in the x-domain (that point needs not belong to the defined domain), with the following property: if we consider however small a neighbourhood of that point, and consider the values of y corresponding to that x-domain, then these values of y also have an upper bound, this upper bound being exactly g. Similarly for the lower bound.
>
> (...) One is commonly faced with the question: among the values taken on by some magnitude, is there a maximum or a minimum (maximum or minimum in the absolute sense). Let y be a continuous function of x, $y = f(x)$. Here, x must remain between two given limits a and b. In which circumstances is there a maximum and a minimum for y ? There is an upper bound for y. According to our proposition, there must be some point x_0 in the x-domain such that the upper-bound of the values of y for x between $x_0 - \delta$ and $x_0 + \delta$ is also g. Point x_0 either lies inside $a...b$, or on its border ($x_0 = a$, or $x_0 = b$).
>
> In the first case, $f(x_0)$ is a maximum. Indeed, $f(x_0)$ must be equal to g: for $f(x)$ - $f(x_0)$ can be made as small as we wish, by choosing an adequately small $|x - x_0|$; on the other hand, since x lies between $x_0 - \delta$ and $x_0 + \delta$, $f(x)$ can be chosen arbitrarily close to g; hence $f(x_0) = g$. (If we had $f(x_0) = g+h$, we would have $f(x) - f(x_0) = f(x) - g - h$, and $f(x)$ could not come arbitrarily close to g if h was not 0).
>
> If x_0 coincided with either a or b, then we could only claim that $f(a)$ (resp. $f(b)$ is a maximum if $f(x)$ was continuously at a (resp. b) as well. (Weierstrass, 1988, p.91-92)

DIDACTICAL PERSPECTIVES

Epistemological summary

First, let us attempt to summarize the pretty intricate network of definitions, proof-ideas (or proof-germs (Downs & Mamona-Downs, 2010)), proof-techniques, and proof-analyses displayed in this sample of texts. What follows constitutes an *epistemological narrative*.

At least two *definitions* of what it means for a real-valued function to "increase" can be found in the 19th-century: a point-wise and global definition which can be found in Lagrange; a definition that relies on an everywhere-valid local property, which can be found in Cauchy. If we stick to Cauchy's definition, then the proof of the theorem about the relationship between the sign of *f'* and the variations of *f* is pretty trivial. If we want to reach the Lagrange-style conclusion, then much more work is needed, since one has to start from an everywhere-valid local property (sign of *f'*) and reach a global conclusion.

To reach that conclusion, we saw two very different *proof-ideas*, namely Lagrange's and Bonnet's. In the proof we studied, and in quite a few other parts of his work, Lagrange distanced himself from the formal manipulation of formulae (finite or infinite), and engaged in numerical proof: he relied on the correct numerical understanding of the notion of limit; on this basis, he cautiously built networks of inequalities; he finally endeavoured to ground his reasoning on the determination of upper bounds for the errors in a process of affine approximation. In the first half of the 19th century, many proofs of the most important theorems in function theory were written along this line. Distrust of this proof-scheme spread as mathematicians grew aware of the distinction between point-wise and uniform (continuity, convergence). They spread all the more slowly since the theorems were correct, the building blocks of the proofs showed a proper understanding of the notions at stake, and local counterexamples[7] were hard to find. As Darboux insightfully (but to no avail, as far as Hoüel was concerned) stressed, there were only two ways out of this predicament: either to change proof-germs, or to establish uniformity.

For the theorem on which we chose to focus, an alternative proof became available in the 1860s, which relied on a completely different proof-idea; unlike Lagrange's proof, it did not rely on what the derivative of a function at a point *is* (a limit, which provides some local affine approximation), but on a *property* of the derivative (stated in the mean value theorem). Some elements of Bonnet's proof were later seen as insufficiently grounded, in particular the existence of a minimum or a maximum; in the 1890s, mathematicians such as Jordan used Weierstrass' analysis of the set-theoretic properties of the real line to back up that weaker step in Bonnet's proof.

Didactical issues, and topics to be discussed

As to connections with current work on mathematical proof from a maths education research perspective, we would like to stress several features; we hope this will trigger discussion, possibly collaboration:

* Instead of focusing on *one* proof, the (abridged) corpus we presented and the epistemological narrative we based upon it deal with a *chain of proofs* (or *deductive* chain) which has remained stable since the beginning of the 20th century. The strong deductive nature of the final chain, with it seemingly necessary conceptual connections, makes it difficult to even imagine how it could have emerged progressively. The corpus shows how this emergence is (1) a collective phenomenon (a point emphasized in (Balacheff, 1987, p.148)), (2) gradual. As to the second point, it turns out that the various stages do not make up a linear chain of rigorization steps, in which each tentative proof would be more sea-worthy than the previous one. Indeed, the stages are of a much more varied epistemological nature, and do not make

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[7] Meaning: a counterexample to a step in a proof, not to the theorem itself.

up one single line: really *different* proof ideas were at play; *several* forms of rigorization can be encountered, such as regressive analysis (finding conceptual grounds from which to derive something that was previously taken to be obvious or improvable), or conceptual distinction (between point-wise and uniform). From a teaching perspective, we feel the use of such corpuses in teacher-training could help change their *image* of maths (i.e. their view of the *nature* of mathematics). We should also discuss whether or not *meta*-level, epistemological, descriptive terms such as those we used[8] should/could be used in teacher-training.

* Here, the emphasis lies on the *reading* of proofs rather on the writing of proofs. Hence, this corpus and this narrative do not directly address central issues such as the distinction between argumentation and proof, or the question of what it takes to write a formal proof-text on the basis of a well-grounded conjecture or even a proof-idea. We focused on the analysis of proof-texts, in terms of rigour and conceptual content (i.e. concepts at play); an analysis carried out by mathematicians; an analysis which students could be asked to carry out[9].

* In the corpus we presented, we did not lay the emphasis on the use (or misuse, or lack of use) of quantifiers. It would have been possible; it would have been relevant. We focused on conceptual content rather than on deductive rigour, and claim it leads to fruitful questions. For instance, we would like to know to what extent students are able to recognize/identify fundamental theorems or concepts, when they are stated in a slightly unusual form (think of Lagrange's wording of the theorem, or his and Cauchy's use of the notion of derivative-as-a-limit); we would like them to explore whether or not, and in what respect, Lagrange's and Cauchy's views on what "increasing function" means are equivalent.

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[8] Among others: local counter-example, conceptual distinction, regressive analysis, proof-germ, in-action definition; in other texts, we relied heavily on: concept image / concept definition, change of semiotic registers etc. (Chorlay, 2007).

[9] This potential *task* was already mentioned in the conclusion of Robert and Schwartzenberger's survey paper of 1991, among tools to enhance students' proof-writing and proof-understanding skills: "Thirdly, one can suggest instruction based upon the activities of mathematicians themselves, for example through the study of historical mathematical texts. A difficulty here is the barrier of notation and language as well as the extreme difficulty of many concepts when formulated in their original contexts." (Robert & Schwartzenberger, 1991). Although the difficulties mentioned at the end of the quote are usually quite overwhelming, we would like to argue that it is not the case here.

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