USE OF FORMALISM IN MATHEMATICAL ACTIVITY

CASE STUDY: THE CONCEPT OF CONTINUITY IN HIGHER EDUCATION

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In this paper, we take an interest on logical-mathematical formalism in mathematical statements and we examine the question of formalism through a concept worked in higher education whose logical structure is complex, it is the notion of continuity. The choice of this concept is based on the fact that this concept involves a large number of related variables and that its logical structure is relatively simple because all quantifiers are at the top of the form (Chellougui, 2009).

We present in first few elements which are the didactic transposition of the continuity from knowledge learned to knowledge teach. In a second step, we are interested in the definition of continuity as presented in various mathematics textbooks for first year university science section.

INTRODUCTION

Tunisian secondary education, according to official instructions, logical symbols (logical connectors: $\Rightarrow$, $\Leftrightarrow$..., and quantifiers: $\forall$, $\exists$) are not introduced, and mathematical statements (theorems, definitions) are generally expressed in natural language (Chellougui, 2009). However, from the beginning of the first year of university scientific formalized statements are introduced without a specific work on the operating rules of the symbolism is conducted and without the passage of the statements in natural language to formalized statements is worked. This introduction is motivated by the supposed superiority in terms of operating statements fully or partially formalized. This formalism seems to be an obstacle to mathematical work and therefore acquisitions in the conceptualization (Quine 1970). Thus, in mathematical activity in the first year of university, we identify the problems of interpretation of logical-mathematical vocabulary or gap of operating order to students, specifically difficulties in manipulation of complex statements with multiple quantifications. Generally, these issues are not worked in used textbooks. We adopted the assumption that they reflect ordinary mathematical practice of mathematics teachers (Durand-Guerrier, 2005).

To illustrate this, we chose a concept studied in high school and again in college, that of continuity. For this concept, the question of passage of the definition in natural language to a formal definition is asked. Here, for example, the definition of continuity of a function at any point proposed by Schwartz (1991):

> (For all $a\in\mathbb{R}$) (for all $\varepsilon>0$) (there exist $\eta>0$ such that) (for all $x\in\mathbb{R}$ such that $|x-a|\leq\eta$), we have: $|f(x)-f(a)|\leq\varepsilon$. This sentence can be written formally. [...]:
> $(\forall a\in\mathbb{R})\ (\forall\varepsilon>0)\ (\exists\eta>0)\ (\forall x\in\mathbb{R})\ [(|x-a|\leq\eta)\Rightarrow|f(x)-f(a)|\leq\varepsilon]$. (p.20)

Here, there are three universal quantifiers and an existential quantifier in third position. The scope of these quantifiers on implication is between parentheses. The study proposed here concerns the definition of the term as found in several mathematical works of first year university science section. Prior to this study, we present some elements which are the didactic transposition of the continuity of knowledge savant to knowledge to be taught.

SOME ELEMENTS OF DIDACTIC TRANSPOSITION

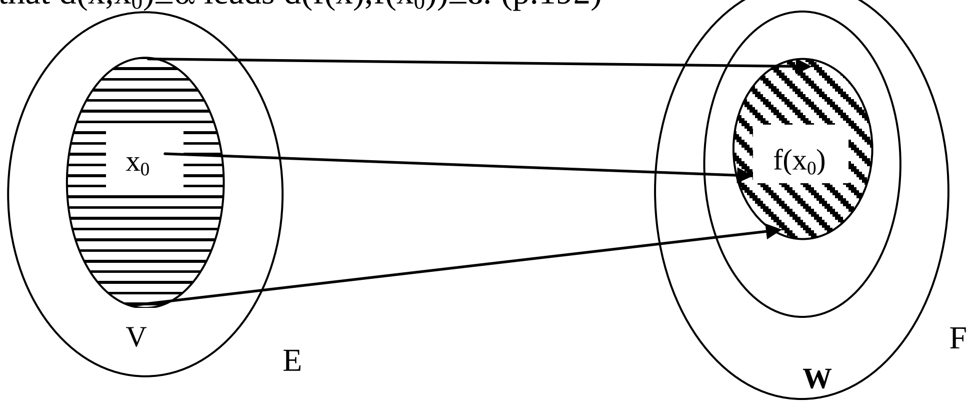
The first comprehensive outline of the Didactic Transposition Theory was developed in Chevallard (1991). The theory is aimed at producing a scientific analysis of didactic systems and is based on the assumption that the mathematical knowledge set up as a teaching object ('savoir enseigné'), in an institutionalised educational system, normally has a preexistence, which is called scholarly knowledge ('savoir savant').

Some objects of scholarly mathematical knowledge are defined as direct teaching objects and constructed in the didactic system (by definition or construction), i.e. mathematical notions, such as for example addition, the circle, or second order differential equations with constant coefficients. However, there are other knowledge objects, termed para-mathematical notions, useful in mathematical activities but often not set up as teaching objects per se but pre-constructed, such as the notions of parameter, equation, or proof (Klisinska, 2009).

We try to analyze the question of the use of para-mathematical logical symbolism in mathematical activity.

We start with the general definition of continuity of a function at a point made in a Dictionary of Mathematics (Bouvier and al., 1979):

> *Application continuous at a point*. – An application f of a topological space E into a topological space F is continuous at $x_0\in E$ if for all neighborhoods W of $f(x_0)$ in F, there exists neighborhood V of x_0 in E whose image f(V) is contained in W. This is the mathematical expression of the sentence "$f(x)$ tends to $f(x_0)$ as x tends to x_0". In the case where E and F are metric spaces, f is continuous at $x_0\in E$ if any $\varepsilon>0$, there exists $\alpha>0$ such that $d(x,x_0)\leq\alpha$ leads $d(f(x),f(x_0))\leq\varepsilon$. (p.192)



(p.192)

This definition is given in topological spaces and neighborhoods; it is then translated into metric spaces. It is formulated in a mixed language supported by the natural and with mathematical symbols. The authors do not offer any logical symbol but illustrate this definition by charts. They were inspired by the approach of Bourbaki (1971), which itself offers the following definition:

> Definition 1. - We say that f of a topological space X into a topological space X is continuous at a point $x_0 \in X$ if any neighborhood V' of $f(x_0)$ in X', there exists a neighborhood V of x_0 in x such that $x \in V$ implies $f(x) \in V'$. (P.I.8)

While, to define a uniformly continuous application, Durand-Guerrier and Arsac (2003) call for logical symbolism and even offer a fully formal definition:

> Application uniformly continuous. – An application f of a metric space E with values in a metric space F is uniformly continuous if for all $\varepsilon>0$, there exists $\eta(\varepsilon)>0$ such that $d(x,y)\leq\eta(\varepsilon)$ implies $d(f(x),f(y))\leq\varepsilon$. This can be symbolized by:
> $(\forall\varepsilon>0)\ (\exists\eta(\varepsilon)>0)\ (\forall(x,y)\in E^2)\ (d(x,y)\leq\eta(\varepsilon) \Rightarrow d(f(x),f(y))\leq\varepsilon)$. (p.193)

In this definition the quantification is complete and there is the appearance of dependency relationship η versus ε.

To give a definition of continuity in $\mathbb{R}$, it is possible to restrict the use of intervals, we get a definition equivalent to that set out above, since in effect, the intervals form a basis of neighborhood in set $\mathbb{R}$.

This is what proposes Haug (2000):

> **Definition 6.a** Let *E* be a set of real numbers. Let *f* be an application of *E* into IR, *a* and *b* are real numbers.
> We say that *b* is a *limit* of *f* if: any open interval J centred on *b*, there exists an open interval I centred on *a* such that $f(E\cap I)\subset J$. (p.107)

He then notes:

> Show that if we replace the above equation by the following equation we obtain an equivalent definition.
> $\forall_{\mathbb{R}_+^*}\ \varepsilon,\ \exists_{\mathbb{R}_+^*}\ \eta,\ \forall_E\ t, |a-t| \leq \eta \Rightarrow |b-f(t)| \leq \varepsilon$. (p.107)

Here, we note that there are no parentheses to express the scope of quantifiers on implication fairly common practice among authors of textbooks and mathematicians. Some students may not be aware of the difficulties instead of quantifiers and parentheses. It is indeed important to understand the effects on the meaning of a statement and interpret in a mathematical theory, when changing the order of quantifiers (Dubinsky & Yiparaki, 2000).

Farther, we read:

> **Definition 6.b**– Let *E* be a set of real, let *a* be an element of *E*, let *f* an application from *E* to $\mathbb{R}$.
> We say that *f* is *continuous* at *a* if *f* has a limit at *a*. (p.111)

The limit definition is followed by a geometric representation based primarily on intervals of IR, which can illumine the situation. There is an explanation of the passage: neighborhood-distance that is not very common in other textbooks studied. The intervals favour the didactic transposition of the definition with the neighborhoods, they reduce the number of quantifiers in the game and can in some cases be easier to handle (Chellougui, 2009).

STUDY SOME TEXTBOOKS: CONCEPT OF CONTINUITY IN THE KNOWLEDGE TO BE TAUGHT

We present below a study of certain textbooks for students in their first year of university. We chose these books because their use by students and teachers of the Faculty of Sciences of Bizerte. Textbooks examined are: Chambadal and Ovaert[1], *Mathematics*, Volume 1: Basic concepts of algebra and analysis, 1966; Arnaudies and Fraysse[2], Mathematics-2-Analysis, 1st cycle university preparatory classes-, 1988; Schwartz[3], Analysis-I, Set theory and Topology-1991; Guégand and Gavini[4], HEC-Analysis-prepa H.E.C-,1995.

Our focus is on the different types of language used. We emerged from this study three phenomena that we analyze below.

Implication versus bounded quantification

In a mathematics textbook (Chambadal & Ovaert, 1966), we find the limit definition of a function at a point followed by that of continuity.

> **Definition 19. - Limit of a function at a point**. - Let *f* be a function defined on a part A of $\overline{\mathbb{R}}$ and x_0 an accumulation point of A. We say that *f* has a limit at x_0 if it has a limit when x tends to x_0 remaining in $P=A-\{x_0\}$. We also say more briefly that *f* has a limit when x tends to x_0. (pp.394-395)

This definition is given in natural language, then explicit and formalized:

> -If x_0 is finite, so that f tends towards l when x tends to x_0, it is necessary and sufficient that:
> $\forall \varepsilon \in \mathbb{R}_+^*, \exists\ \eta \in \mathbb{R}_+^* : \forall x \in A \cap ([x_0-\eta, x_0+\eta]-\{x_0\}), |f(x) - l| \leq \varepsilon$ (1)[5]
> what writes:
> $\forall \varepsilon \in \mathbb{R}_+^*, \exists\ \eta \in \mathbb{R}_+^* : \forall x \in A, |x-x_0| \leq \eta$ et $x \neq x_0 \Rightarrow |f(x) - l| \leq \varepsilon$. (2) (pp.395-396)

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[1] Chambadal et Ovaert, *Cours de mathématiques –Tome 1 : notions fondamentales d'algèbre et d'analyse–*, 1966

[2] Arnaudiès et Fraysse, *Cours de mathématiques-2 –Analyse, Classes préparatoires 1^er cycle universitaire–*, 1988

[3] Schwartz, *Analyse WE–Théorie des ensembles et topologie–*, 1991.

[4] Guégand et Gavini, *Analyse –prépa H.E.C–*, 1995.

[5] To facilitate links between different formal statements, we numbered ourselves those statements from (1) to (12) in the illustrations extracted from the different textbooks or in our own analyses.

A beginning reader might ask where does the implication that appears in (2).

In another analysis textbook of first year science (Guégand & Gavini, 1995), we can learn:

> **2.1 Definition**: Let I be an interval of $\mathbb{R}$, $a \in$ I and f : I$\rightarrow \mathbb{R}$
> We say that f is **continuous at *a*** if and only if f admits f(a) as a limit.
> Otherwise we say that f is discontinuous at a.
> Let us clarify this definition:
> f is continuous at a
> $\Leftrightarrow \quad \forall\varepsilon>0, \exists\ \alpha>0, \forall x\in I, |x-a| < \alpha \ \Rightarrow |f(x)-f(a)| < \varepsilon$ (3)
> $\Leftrightarrow \quad \forall\varepsilon>0, \exists\ \alpha>0, \forall x\in I, |x-a| \leq \alpha \ \Rightarrow |f(x)-f(a)| \leq \varepsilon$ (4)
> $\Leftrightarrow \quad \forall\varepsilon>0, \exists\ \alpha>0, \forall x\in I\cap [a-\alpha,a+\alpha]\ , |f(x)-f(a)| \leq \varepsilon.$ (5) (p.108)

The beginning reader might here wonder where the implication that disappears (5). This game appearance/disappearance of implication is related to mathematical practice of bounded quantification. This type of quantification is present in mathematics, but absent in the predicate calculus. For example, the mathematical writing:
$\forall x\in A\ F(x)$ is reflected in the predicate calculus by: $\forall x(x\in A \Rightarrow F(x))$.

It is bounded quantification hides the implication. The reference domain in the first write is limited, then it is expanded in the second. Instead of quantifying over elements that belong to $\mathbb{R}$ or A, we quantified on elements that satisfy the antecedent of the conditional, which removes the implication: the practice of bounded quantification is present in textbooks, several authors give a formulation without the conditional.

We speak of bounded quantification to describe a common practice when the mathematician wants to give a definition on formal language. Which appears in both textbooks is that Chambadal, Ovaert, Guégand and Gavini use this practice without problematized without the work. But Durand-Guerrier (2005) underlines that the implication is not restored naturally by students, whereas, this constitutes the sense of implication.

Implication versus conjonction

In another textbook of mathematics (Arnaudies & Fraysse, 1988), the authors give first continuity at a point with the neighborhood in a metric space:

> Definition III.4.1 - Let A be a part of $\mathbb{R}$ and f: A $\rightarrow \mathbb{R}$ a function. We say that f is continuous at $a\in$A if and only if for every neighborhood W of $f(a)$ there exists a neighborhood V of a such that $f(V\cap A)\subset W$.
> We say that f is discontinuous at point $a\in$A if and only if it is not continuous at that point.
> The function f is continuous if and only if it is continuous at every point of A. (p.108)

After they present the classic definition using a mixed language:

[...] we obtain in particular the following definitions of continuity of f at a equivalent to the definition III.4.1:

(I) For all real $\varepsilon>0$, there exists a real $\eta>0$ such that
$(x\in A \text{ and } |x-a|\leq\eta)\Rightarrow(|f(x)-f(a)|\leq\varepsilon)$

(II) For all real $\varepsilon>0$, there exists a real $\eta>0$ such that
$(x\in A \text{ and } |x-a|<\eta)\Rightarrow(|f(x)-f(a)|<\varepsilon)$. (p.108)

We can ask: why is there "and" in the antecedent of the implication?
We rather expect the following entry:
For all real $\varepsilon>0$ there exists a real $\eta>0$ such that for all $x\in A$ $(|x-a|<\eta)\Rightarrow(|f(x)-f(a)|<\varepsilon$
or:

$$\forall\ \varepsilon>0\ \ \exists\ \eta>0\ \ \forall x\in A\ \ (\,|\,x-a\,|<\eta) \Rightarrow (\,|\,f(x)-f(a)\,|<\varepsilon) \qquad (6)$$

Another formulation of the statement (II) presented in the textbook, using logical symbols for the quantification of each variable ε and η gives:

$$\forall\ \varepsilon>0\ \ \exists\ \eta>0\ \ (x\in A\ \text{ and }\ |\,x-a\,|<\eta) \Rightarrow (\,|\,f(x)-f(a)\,|<\varepsilon\,) \qquad (7)$$

There it equivalence between writing (6) and (7)?

As noted above, removing the bounded quantification on the variable x to the statement (6), we obtain: $\forall\varepsilon>0\ \exists\eta>0\ \forall x\ [x\in A \Rightarrow (|\,x-a\,|<\eta\Rightarrow|\,f(x)-f(a)\,|<\varepsilon)]$.
Our question back down to logical equivalence between:

$$(x\in A\ \wedge\ |\,x-a\,|<\eta) \Rightarrow (\,|\,f(x)-f(a)\,|<\varepsilon\,) \qquad (8)$$

and $[x\in A \Rightarrow (\,|\,x-a\,|<\eta \Rightarrow |\,f(x)-f(a)\,|<\varepsilon\,)]$ (9)

Considering only the variable x, the set (8) and (9) are of the form:
$(p(x)\wedge q(x)) \Rightarrow r(x)$ and $[p(x) \Rightarrow (q(x)\Rightarrow r(x))]$.

However, in the propositional calculus, the following equivalence:
$[p \Rightarrow (q\Rightarrow r)] \Leftrightarrow [(p\wedge q) \Rightarrow r]$ is a tautology.

Where equivalence in predicate calculus:
$\forall x\ [p(x) \Rightarrow (q(x)\Rightarrow r(x))] \Leftrightarrow [(p(x)\wedge q(x)) \Rightarrow r(x)]$ is universally valid.

So using logical arguments, we prove the equivalence between the two statements: (6) and (7). We summarize this equivalence in the following table with a justification in logical syntax by translating the writing mathematics in the predicate calculus (Kouki, 2008):

Writing mathematical	Predicates calculus
$\forall x\in A\ (\ F(x) \Rightarrow G(x)\ )$	$\forall x\ [\ x\in A \Rightarrow (F(x) \Rightarrow G(x))] \equiv$ $\forall x\ [\ (x\in A \wedge F(x)) \Rightarrow G(x)\ ]$
$\forall x\ [\ (x\in A \text{ et } F(x)) \Rightarrow G(x)\ ]$	

Negation of formalized statements

a- One-interest, generally recognized, formal logic is to facilitate the transition to negation. This is what is written, for example, by Guégand and Gavini(1995):

> *Those quantified expressions* can be easily denied. So to formulate that f does not tend to L (real) at a (real), we have (negation of the definition)
> $\exists\varepsilon>0,\ \forall\eta>0,\ \exists x\in U, |x-a|<\eta \quad |f(x)-L|\geq\varepsilon$ (10) (p.108)

In the textbook *those quantified expressions* mean:

> (1') $\forall\varepsilon>0,\ \exists\eta>0,\ \forall x\in U, |x-a|<\eta \Rightarrow |f(x)-L|<\varepsilon$
> (2') $\forall\varepsilon>0,\ \exists\eta>0,\ \forall x\in U, |x-a|\leq\eta \Rightarrow |f(x)-L|\leq\varepsilon$
> (3') $\forall\varepsilon>0,\ \exists\alpha>0,\ \forall x\in U\cap[a-\alpha, a+\alpha], |f(x)-L|\leq\varepsilon$. (p.107)

In the statement (10), a blank is detected between the two inequalities: ($|x-a|<\eta$) and ($|f(x)-L|\geq\varepsilon$). How to fill this blank? It is the negation of (1'), (2') or (3')?
The negation of (1') is: $\exists\varepsilon>0,\ \forall\eta>0,\ \exists x\in U, |x-a|<\eta \ \wedge\ |f(x)-L|\geq\varepsilon$
The negation of (2') is: $\exists\varepsilon>0,\ \forall\eta>0,\ \exists x\in U, |x-a|\leq\eta \ \wedge\ |f(x)-L|>\varepsilon$
The negation of (3') is: $\exists\varepsilon>0,\ \forall\eta>0,\ \exists x\in U\cap[a-\alpha, a+\alpha]\ |f(x)-L|>\varepsilon$

The expression (10) doesn't correspond to any of the previous negations. We hypothesize that the authors thought of negation of written (1) and they do not want to use a logical symbolism for the conjunction "and". If they did not put the word "and" in this blank is to keep all the words in formal language and do not use a mixed language. This is based on the fact that in mathematics we used very rarely logical symbol "$\wedge$" which translates the conjunction. This is reflected in the textbooks studied. Indeed, the logic symbol of the conjunction is identified once among the textbooks studied in the first part entitled: *Set Theory*, of Laurent Schwartz (1991) and in the first paragraph entitled *Some elements of classical logic*. The author, in this section, using the definition of continuity of a real function to illustrate the rules for handling negation: inversion of two types of quantification, negation of implication. We can read:

> For express now that the function is continuous at every point, we write:
> […]
> "$(\forall a\in\mathbb{R})\ (\forall\varepsilon>0)\ (\exists\eta>0)\ (\forall x\in\mathbb{R})\ [(|x-a|\leq\eta) \Rightarrow |f(x)-f(a)|\leq\varepsilon]$ (p.20)

Further, we read:

> For example, the property for a function f of a real variable not is everywhere continuous, that is to say to be discontinuous at least one point is expressed by the single line:
> $(\exists a\in\mathbb{R})\ (\exists\varepsilon>0)\ (\forall\eta>0)\ (\exists x\in\mathbb{R})\ [(|x-a|\leq\eta) \wedge (|f(x)-f(a)|>\varepsilon)]$ (p.21)

This statement is obtained by recursively applying transformation rules:
$\neg(\forall x\ Fx) \equiv \exists x\ \neg Fx$ (11)
$\neg(\exists x\ Fx) \equiv \forall x\ \neg Fx$ (12)

which allow to focus on the open sentence negation with no quantification. The application of the general rule: $\neg(p \Rightarrow q) \equiv p \wedge \neg q$ to open sentences (11) and (12), finally allows to focus only on the negation atomic formula: $|f(x) - f(a)| \leq \varepsilon$. The logical negation symbol disappearing by the equivalence between the relation ">" and the negation of "$\leq$".

Note that this transformation process quantifiers only applies if they are leading formula. Indeed, a quantifier in the antecedent of head involvement is not changed by passing the negation: $\neg[(\forall x\ Fx) \Rightarrow G] \equiv \forall x\ Fx \wedge \neg G$

b-In the textbook Arnaudies and Fraysse (1988) mentioned above, following the definition of continuity, the authors define the discontinuity noting:

> The discontinuity of f at a point $a \in A$ means:
> (III) There exist $\varepsilon > 0$ such that for all $\eta > 0$, ($x \in A$ and $|x - a| \leq \eta$) $\not\Rightarrow$ ($|f(x) - f(a)| \leq \varepsilon$)
> i.e
> (IV) There existe $\varepsilon > 0$ such that for all $\eta > 0$ we can fond at least one x in A such that $|x - a| \leq \eta$ and $|f(x) - f(a)| > \varepsilon$. (p.109)

In the statement (III), there is a use of logic-mathematical symbols do not conform to the syntax of logic, it is $\not\Rightarrow$. It begs the question: what is barred? Especially since it is an implicit universal quantification: [($x \in A$ and $|x - a| \leq \eta$) $\not\Rightarrow$ ($|f(x) - f(a)| \leq \varepsilon$)]

The authors answer in the statement (IV) using a given vocabulary in a language of action, where we expect more writing using the logic symbol of the existential quantifier. This shows that what is barred is naturally involving universally quantified.

Indeed, the following writing: $\forall x$ ($x \in A$ and $|x - a| \leq \eta$) $\not\Rightarrow$ ($|f(x) - f(a)| \leq \varepsilon$) is interpreted by: None x satisfies the implication, which is not the negation of the definition.

The negation follows: $\neg\ [(x \in A \wedge |x - a| \leq \eta) \Rightarrow (|f(x) - f(a)| \leq \varepsilon)]$ does not give rise to the appearance of an existential quantifier.

The negation is obviously on the universal $\neg[\forall x\ (x \in A \wedge |x - a| \leq \eta) \Rightarrow (|f(x) - f(a)| \leq \varepsilon)]$

Mobilization/immobilization quantification product questions about the reading expressions, the use of a mixed formalism gives rise to difficult reading. Thus, there are ambiguities of natural language that formalism supposed to lift.

CONCLUSION

Logico-mathematical formalizations proposed definitions of continuity, limits and discontinuity are different from one textbook to another, one might think they reflect the everyday practices of mathematicians. There seems to be an illusion of transparency in the way of natural language to writing relatively formalized to the extent that this passage is little or no work.

In some expressions, the presence of bounded quantification is indicated; practice creates a phenomenon of appearance/disappearance of involvement and quantification product entries do not conform to the syntax and logic that generates ambiguity, then the transition formalism is supposed ambiguities of ordinary language (Kouki, 2006). In addition, the use of automatic syntactic rules, not problematized, to construct recursively the negation of a sentence obscures many fundamental questions for operative use of formalism (Durand-Guerrier and al., 2012).

The introduction of logical-mathematical formalism in the learned knowledge is to introduce a certain level of mathematical rigor in writing to lift ambiguities, implicit and ask some evidence. In the knowledge to be taught, the study showed a wide variety of formulations formal language, natural or mixed for which we have identified and analyzed syntactic difficulties which we can assume they will affect student work.

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